On the Figure and Stability of a Liquid Satellite.

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(Abstract.)

More than half a century ago Edouard Roche wrote his celebrated paper on the form which a liquid satellite will assume when revolving, without relative motion, about a solid planet.* As far as I know, his laborious computations have never been repeated, and their verification and extension form a portion of the work contained in the present paper.

Two problems involving almost identical analysis, but very distinct principles, are here treated simultaneously. If we imagine two detached masses of liquid to revolve about one another in a circular orbit without relative motion, the determination of the shapes of each of them is common to both the problems; it is in the conditions of their secular stability, according to the suppositions made, that the division occurs.

The friction of the tides raised in each mass by the attraction of the other is one cause of instability. If now the larger of the two masses were rigid, whilst still possessing the same shape as though liquid, the only tides subject to friction would be those in the smaller body. It amounts to exactly the same whether we consider the larger mass to be rigid or whether we consider it to be liquid, and agree to disregard the instability which might arise from the tidal friction of the tides generated in it by the smaller body. Accordingly I describe secular stability in the case just considered as "partial," whilst complete secular stability will involve the tidal friction in each mass.

The determination of the figure and partial stability of a liquid satellite is the problem of Roche. It is true that he virtually considered the larger body or planet to be a rigid sphere, but in this abstract the distinction introduced by the fact that I treat the planet as ellipsoidal may be passed over. It appears that, as we cause the two masses to approach one another, the partial stability of Roche's satellite first ceases to exist through the deformation of its shape, and certain considerations are adduced which show that the most interesting field of research is comprised in the cases where the satellite ranges from infinite smallness relatively to the planet to equality thereto.

The limiting partial stability of a liquid satellite is determined by considering the angular momentum of the system, exclusive of the rotational momentum of the planet. This corresponds to the exclusion of the tidal

* 'Mém. Acad. Sci. de Montpellier,' vol. 1, 1847—50, p. 243.

friction of the tides raised in the planet. For any such given angular momentum there are two solutions, if there is any. When these two solutions coalesce for minimum angular momentum, we have found a figure of bifurcation; for any other larger angular momentum one of the solutions belongs to an unstable series and the other to a stable series of figures. Thus, by determining the figure of minimum partial angular momentum, we find the figure of limiting partial stability.

The only solution for which Roche gave a numerical result was that in which the satellite is infinitesimal relatively to the planet. He found that the nearest possible infinitesimal satellite (which is also in this case the satellite of limiting partial stability) has a radius vector equal to 2·44 radii of its spherical planet. He showed the satellite to have an ellipsoidal figure, and stated that its axes were proportional to the numbers 1000, 496, 469. In the paper the problem is solved by more accurate methods than those used by Roche, and it is proved that the radius vector is 2·4553, and that the axes of the ellipsoid are proportional to 10,000, 5114, 4827. The closeness with which his numbers agree with these shows that he must have used his graphical constructions with great care.

For satellites of finite mass the satellite is no longer ellipsoidal, and it becomes necessary to consider the deformation by various inequalities, which may be expressed by means of ellipsoidal harmonic functions.

The general effect for Roche's satellites of finite mass in limiting partial stability is that the ellipsoidal form is very nearly correct over most of the periphery of the satellite, but at the extremity facing the planet there is a tendency to push forth a protrusion towards the planet. In the stable series of figures up to limiting stability this protrusion is of no great magnitude, but in the unstable series it would become strongly marked. When the unstable figure becomes much elongated, we find that it finally overlaps the planet, but before this takes place the approximation has become very imperfect.

Turning now to the case of complete secular stability, where the tidal friction in each mass is taken into account, we find that for an infinitely small satellite limiting stability occurs when the two masses are infinitely far apart. It is clear that this must be the case, because a rotating liquid planet will continue to repel its satellite so long as it has any rotational momentum to transfer to orbital momentum through the intervention of tidal friction. Thus an infinitesimal satellite will be repelled to infinity before it reaches the configuration of secular stability. As the mass of the satellite increases, the radius vector of limiting stability decreases with great rapidity, and for two equal masses, each constrainedly spherical, the configuration is reached when the radius vector is 2·19 times the radius of either body.

When we pass to the case where each liquid mass is a figure of equilibrium, the radius vector for limiting stability is still infinite for the infinitely small satellite, and diminishes rapidly for increasing mass of the satellite. When the two masses are equal the radius vector of limiting stability is 2·638 times the radius of a sphere whose mass is equal to the sum of the masses of the two bodies. This radius vector is considerably greater than that found in the case of the two spheres, for the 2·19 radii of either sphere, when expressed in the same unit, is only 1·74. Thus the deformations of the two masses forbid them to approach with stability so near as when they were constrainedly spherical.

In all these cases of true secular stability, instability supervenes through tidal friction, and not, as in the case of Roche's problem, through the deformation of figure.

When Poincaré announced that there was a figure of equilibrium of a single mass of liquid shaped something like a pear, he also conjectured that the constriction between the stalk and the middle of the pear would become developed until it was a thin neck; and yet further that the neck might break and the two masses become detached. The present revision of Roche's work was undertaken in the hope that it would throw some light on the pear-shaped figure in the advanced stage of development.

As a preliminary to greater exactness, the equilibrium is investigated of two masses of liquid each constrainedly spherical, joined by a weightless pipe. Through such a pipe liquid can pass from one mass to the other, and it will continue to do so until, for given radius vector, the masses of the two spheres bear some definite ratio to one another. In other words, two spherical masses of given ratio can be started to revolve about one another in a circular orbit, without relative motion, at such a distance that liquid will not pass through a pipe from one to the other.

The condition for equilibrium is found to be expressible in the form of a cubic equation in the radius vector, with coefficients which are functions of the ratio of the masses. Only one of the three roots of the cubic has a physical meaning, and in all cases the two masses are found to be very close together; but the system can never possess secular stability.

When the masses are no longer constrainedly spherical the equation of condition for equilibrium, when junction is effected by a weightless pipe, becomes very complicated and can only be expressed approximately. It appears that in all cases, even of Roche's ellipsoids in limiting stability, the masses are much too far apart to admit of junction by a pipe; but when we consider the unstable series of much elongated ellipsoids, it seems that such junction is possible, although the approximation is too imperfect to enable us to draw the figure with any approach to accuracy. If two ellipsoids are unstable when moving detached from one another, junction by a pipe cannot possibly make them stable. This then points to the conclusion that the pear-shaped figure is unstable when so far developed as to be better described as two bulbs joined by a thin neck.

Mr. Jeans has considered the equilibrium and stability of infinite rotating cylinders of liquid.* This is the two-dimensional analogue of the three-dimensional problem. He found solutions perfectly analogous to Maclaurin's and Jacobi's ellipsoids and to the pear-shaped figure, and he was able to follow the development of the cylinder of pear-shaped section until the neck joining the two parts had become quite thin. The analysis, besides, points to the rupture of the neck, although the method fails to afford the actual shapes and dimensions in this last stage of development. He is able to prove conclusively that the cylinder of pear-shaped section is stable, and it is important to note that he finds no evidence of any break in the stability up to the division of the cylinder into two parts.

The stability of Maclaurin's and of the shorter Jacobian ellipsoids is well established, and I imagined that I had proved that the pear-shaped figure with incipient furrowing was also stable. But M. Liapounoff† now states that he is able to prove the pear-shaped figure to be unstable from the beginning. For the present at least I still think it is stable, and this belief receives powerful support from Mr. Jeans' researches.

But there is another difficulty raised by the present paper. I had fully expected to obtain an approximation to a stable figure consisting of two bulbs joined by a thin neck, but although the present work indicates the existence of such a figure, it seems conclusive against its stability. If then Mr. Jeans is right in believing in the stable transition from the cylinder of pear-shaped section to two detached cylinders, and if I am now correct, the two problems must part company at some undetermined stage. M. Liapounoff will no doubt contend that it is at the beginning of the pear-shaped series of figures, but for the present I should dissent from that view.

One question remains: If the present conclusions are right, do they entirely destroy the applicability of this group of ideas to the explanation of the birth of satellites or of double stars? I think not, for we see how a tendency to fission arises, and it is not impossible that a period of turbulence may naturally supervene in the process of separation. Finally, as Mr. Jeans points out, heterogeneity introduces new and important differences in the conditions.

* 'Phil. Trans.,' A, vol. 200, pp. 67—104.

† 'Acad. Imp. des Sci. de St. Pétersbourg,' vol. 17, No. 3, 1905.