XI. "On the Harmonic Analysis of Tidal Observations of High and Low Water." By G. H. DARWIN, F.R.S., Plumian Professor and Fellow of Trinity College, Cambridge. Received June 17, 1890.

§ 1. *Introduction.*

Extensive use of the tide-gauge has only been made in recent years, and by far the largest number of tidal records consist only of observations of high and low water (H. and L.W.). Such observations have usually been reduced by determining the law governing the relationship between the times and heights of H. and L.W. and the positions of the moon and sun. This method is satisfactory so long as the diurnal inequalities are small, but it becomes both complex and unsatisfactory when the diurnal inequality is large. In such cases the harmonic notation for the tide is advantageous, and as, except in the North Atlantic Ocean, the diurnal inequality is generally considerable, a proper method of evaluating the harmonic constants from H. and L.W. observations is desirable.

The essential difference between the method here proposed and that followed by Laplace and his successors is that they introduced astronomical considerations from the first and applied them to each H. and L.W., whereas the positions of the sun and moon will only be required here at a single instant of time. In their method, the time of moon's transit, and hence the interval, was found for each tide; the age of the moon, and the moon's and sun's parallaxes and declinations were also required. An extensive table from the astronomical ephemeris was thus necessary, and there still remained the classification of heights and intervals according to the age of moon, and two parallaxes, and two declinations. The classification could hardly be less laborious, and was probably less mechanical, than the sorting processes employed below. There is probably, therefore, a considerable saving of labour in the present method, and, besides, I conceive that the results are more satisfactory when expressed in the harmonic notation.

My object has been to make the whole process a purely mechanical one, and, although nothing can render the reduction of tidal observations a light piece of work, I believe that it is here presented in a form which is nearly as short as possible.

The analytical difficulties to be encountered in such a task are small, but the arrangement of a heavy mass of arithmetic, so as to involve a minimum of labour and therefore of expense, is by no means easy. How far I have succeeded must be left to the decision of those who will, I hope, use the methods here devised.

When a question of this kind is attacked, the solution cannot be deemed complete unless the investigation is left in such a state that an ordinary trained computer is able to use it as a code of instructions by which to reduce a series of observations, without any knowledge of tidal theory.

An actual numerical example is thus essential, both to test the method and to serve as instructions to a computer. The Appendix contains so much of the reduction of three months of observation at Bombay as will serve as such a code. If the series be longer than three months, or in such cases as the proper treatment of gaps in the series, it is necessary to refer back to the body of the paper for instructions.

I now pass to the theoretical reasons for the rules for reduction.

§ 2. *Notation.*

The notation of the Report to the British Association for 1883, and in use in the Indian tidal work and elsewhere, is here followed.

The earth's angular velocity is denoted by γ; the hourly mean motions of the moon, sun, and lunar perigee by σ, η, ϖ (γῆ, σελήνη, ἥλιος); the mean longitudes of moon, sun, and lunar perigee by s, h, p, and the mean solar hour angle by t. The R.A. and longitude in the lunar orbit of the intersection of the equator with the lunar orbit are ν, ξ; and N is the longitude of the moon's node.

The several harmonic tides are denoted by arbitrarily chosen initial letters. Those with which we shall principally have to deal are—

Semi-diurnal.

Name.	Initial.	Speed.	Equilibrium argument
Principal lunar	M_2	$2(\gamma-\sigma)$	$2t+2(h-\nu)-2(s-\xi)$
„ solar	S_2	$2(\gamma-\eta)$	$2t$
Luni-solar....	K_2	2γ	$2t+2(h-\nu'')$
Larger elliptic	N	$2\gamma-3\sigma+\varpi$	$2t+2(h-\nu)-2(s-\xi)-(s-p)$
Smaller „	L	$2\gamma-\sigma-\varpi$	$2t+2(h-\nu)-2(s-\xi)+(s-p)+\pi$

Diurnal.

Name.	Initial.	Speed.	Equilibrium argument
Luni-solar....	K_1	γ	$t+(h-\nu')-\frac{1}{2}\pi$
Lunar	O	$\gamma-2\sigma$	$t+(h-\nu)-2(s-\xi)+\frac{1}{2}\pi$
Solar	P	$\gamma-2\eta$	$t-h+\frac{1}{2}\pi$

The symbol H denotes the mean semi-range of any one of the tides, and κ its retardation of phase behind what it would be according to the equilibrium theory; f denotes a certain factor of augmentation of the lunar and luni-solar tides depending on the value of N.

The particular tide to which H, κ, f refer will in general be indicated by a subscript small letter, the same as the letter constituting the initial of the tide. Thus, for example, the M_2 tide is expressed by

$$f_m H_m \cos(2t + 2(h-\nu) - 2(s-\xi) - \kappa_m).$$

I have allowed a departure from this notation in the case of the tides K_2 and K_1, where I write H'', κ'', f'' for the first, and H', κ', f' for the second. The angles $2\nu''$ and ν' (which, like ν and ξ, are functions of N) are also involved in the arguments* (or angle under the cosine in the expression for the height of the particular tide) of these two tides.

It is obviously necessary to suppose the reader to have some acquaintance with the harmonic notation, or it would be necessary to repeat the Report on Tides above referred to.

§ 3. *The General Method of Treating H. and L.W. Observations.*

Noon of the day on which the observations begin is to be taken as the epoch, and the mean solar time elapsed since epoch is noted by t. V with the proper subscript letter denotes the increase of argument since epoch; for example, $V_m = 2(\gamma-\sigma)t$.

Then the height of the water h, estimated from mean sea-level, is expressed by a number of terms of the form $A\cos V + B\sin V$, or, in an alternative form, $R\cos(V-\zeta)$.

In order to explain the principle of the method proposed, let us take two typical terms involving V_p and V_q, and let the rates of increase of V_p be p, and of V_q be q.

Then we have

$$h = A_p \cos V_p + B_p \sin V_p + A_q \cos V_q + B_q \sin V_q \quad \ldots\ldots \quad (1).$$

Since at H. or L.W. h is a maximum or a minimum, we must have—

$$0 = A_p \sin V_p - B_p \cos V_p + \frac{q}{p} A_q \sin V_q - \frac{q}{p} B_q \cos V_q \quad \ldots\ldots \quad (2).$$

Let us write

$$\frac{q}{p} = k_q \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Then multiply (1) by $\cos V_p$ and (2) by $\sin V_p$, and add; and again multiply (1) by $\sin V_p$ and (2) by $\cos V_p$, and subtract, and we have—

$$\left.\begin{aligned} h\cos V_p &= A_p + A_q(\cos V_p \cos V_q + k_q \sin V_p \sin V_q) \\ &\quad + B_q(\cos V_p \sin V_q - k_q \sin V_p \cos V_q), \\ h\sin V_p &= B_p + A_q(\sin V_p \cos V_q - k_q \cos V_p \sin V_q) \\ &\quad + B_q(\sin V_p \sin V_q + k_q \cos V_p \cos V_q). \end{aligned}\right\} \quad \ldots\ldots \quad (4).$$

* It is well to explain that I have sometimes elsewhere used argument to denote the argument according to the equilibrium theory, that is to say, with κ equal to zero. In this paper I call the latter the equilibrium argument.

Let

$$\left.\begin{aligned} \Sigma &= \tfrac{1}{2}\cos(V_p - V_q) + \tfrac{1}{2}\cos(V_p + V_q) = \cos V_p \cos V_q, \\ \Delta &= \tfrac{1}{2}\cos(V_p - V_q) - \tfrac{1}{2}\cos(V_p + V_q) = \sin V_p \sin V_q, \\ \sigma &= \tfrac{1}{2}\sin(V_p - V_q) + \tfrac{1}{2}\sin(V_p + V_q) = \sin V_p \cos V_q, \\ \delta &= \tfrac{1}{2}\sin(V_p - V_q) - \tfrac{1}{2}\sin(V_p + V_q) = -\cos V_p \sin V_q. \end{aligned}\right\} \quad \ldots\ldots (5).$$

Also let

$$\left.\begin{aligned} \mathrm{F} &= \Sigma + k_q\Delta, & \mathrm{f} &= \sigma + k_q\delta, \\ \mathrm{G} &= -\delta - k_q\sigma, & \mathrm{g} &= \Delta + k_q\Sigma. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6).$$

Then our equations are—

$$\left.\begin{aligned} h\cos V_p &= A_p + \mathrm{F}A_q + \mathrm{G}B_q, \\ h\sin V_p &= B_p + \mathrm{f}A_q + \mathrm{g}B_q. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

A similar pair of equations will result from each H. and L.W. When a series of tides is considered, we may take the mean of the equations and substitute a mean F, G, f, g.

The general principle here adopted is to take the means over such periods that the mean F, G, f, g become very small. In fact, we shall, in several cases, be able to reduce them so far that these terms are negligible, and get simply $\frac{1}{n+1}\Sigma h \frac{\cos}{\sin} V_p = \frac{A_p}{B_p}$; but in other cases, where what is typified as the p tide is a small one, whilst one or more of the tides typified as q is large, it will be necessary to find F, G, f, g. The finding of these coefficients is clearly reducible to the finding of the mean values of $\frac{\cos}{\sin}(V_p \pm V_q)$.

Another useful principle may be illustrated thus: if the q tide does not differ much in speed from the p tide, we may put $V_q = V_p + \nu t$, where ν is a small speed. Then we write

$$\begin{aligned} h &= R_p\cos(V_p - \zeta_p) + R_q\cos(V_p + \nu t - \zeta_q) \\ &= \cos V_p\{R_p\cos\zeta_p + R_q\cos(\nu t - \zeta_q)\} \\ &\quad + \sin V_p\{R_p\sin\zeta_p - R_q\sin(\nu t - \zeta_q)\}. \end{aligned}$$

If we neglect ν/p, the condition for maximum and minimum in conjunction with this gives

$$\begin{aligned} h\cos V_p &= R_p\cos\zeta_p + R_q\cos(\nu t - \zeta_q), \\ h\sin V_q &= R_p\sin\zeta_p - R_q\sin(\nu t - \zeta_q). \end{aligned}$$

Then taking the mean of these equations over a period beginning with $t = 0$ and ending when $t = \pi/\nu$, we have (writing $A_p = R_p \cos \zeta_p$, $B_p = R_p \sin \zeta_p$)

$$\frac{1}{n+1}\Sigma h \cos V_p = A_p + \lambda R_q \cos(\alpha - \zeta_q),$$

$$\frac{1}{n+1}\Sigma h \sin V_p = B_p - \lambda R_q \sin(\alpha - \zeta_q),$$

where λ and α are certain constants, depending on the sum of a trigonometrical series.

Again, if we take means from $t = \pi/\nu$ to $t = 2\pi/\nu$, the second terms have their signs changed.

Hence the difference between these two successive sums will give $\lambda R_q \cos(\alpha - \zeta_q)$ and $\lambda R_q \sin(\alpha - \zeta_q)$. There will be usually two terms such as those typified by q, and we shall then have to take two other means, viz., one beginning at $\pi/2\nu$ and ending at $3\pi/2\nu$, and the other beginning at $3\pi/2\nu$ and ending at $5\pi/2\nu$. From the difference of these sums we get $-\lambda R_q \sin(\alpha - \zeta_q)$ and $\lambda R_q \cos(\alpha - \zeta_q)$. From these four equations the two R_q's and the two ζ_q's are found. The solution is a little complicated in reality by the fact that it is not possible to take $t = 0$ exactly at the beginning of the series, because the first tide does not occur exactly at noon, but this is a detail which will become clear below.

When all the A's and B's or R's and ζ's have been found, the position of the sun and moon at the epoch, found from the Nautical Almanac, and certain constants found from the Auxiliary Tables in Baird's 'Manual of Tidal Observations,'* are required to complete the evaluation of the H's and κ's.

The details of the processes will become clear when we consider the various tides.

It may be worth mentioning that I have almost completely evaluated the F's and G's, which give the perturbation of one tide on another, in the case considered in the Appendix. Without giving any of the details of the laborious arithmetic involved, it may suffice to say that the conclusion fully justifies the omission of all those terms, which are neglected in the computation as presented below.

§ 4. *The tides* N *and* L.

These are the two lunar elliptic tides.

For the sake of brevity all the tides excepting M_2, N, L are omitted from the analytical expressions.

Since $V_n = V_m - (\sigma - \varpi)t, \quad V_l = V_m + (\sigma - \varpi)t,$

* Taylor and Francis, Fleet Street, 1886.

the expression becomes

$$h = A_m \cos V_m + B_m \sin V_m + R_n \cos [V_m - (\sigma - \varpi)t - \zeta_n] + R_l \cos [V_m + (\sigma - \varpi)t - \zeta_l],$$

$$= \cos V_m \{A_m + R_m \cos [(\sigma - \varpi)t + \zeta_n] + R_l \cos [(\sigma - \varpi)t - \zeta_l]\}$$
$$+ \sin V_m \{B_m + R_m \sin [(\sigma - \varpi)t + \zeta_n] - R_l \sin [(\sigma - \varpi)t - \zeta_l]\}.$$

Hence, taking into account the equation which expresses that h is a maximum or minimum, and neglecting the variation of $s-p$ compared with that of V_m, we have—

$$\left.\begin{aligned} h \cos V_m &= A_m + R_n \cos [(\sigma - \varpi)t + \zeta_n] + R_l \cos [(\sigma - \varpi)t - \zeta_l], \\ h \sin V_m &= B_m + R_n \sin [(\sigma - \varpi)t + \zeta_n] - R_l \sin [(\sigma - \varpi)t - \zeta_l] \end{aligned}\right\} \ldots\ldots (8).$$

The mean interval between each tide and the next is 6·210 hours. Then if e be the increment of $s-p$ in that period (so that with $\sigma - \varpi$ equal to 0°·54437 per hour, e is equal to 3°·3807), and if a, b be the values of $(\sigma - \varpi)t + \zeta_n$ and $(\sigma - \varpi)t - \zeta_l$ at the time of the first tide under consideration, the equations corresponding to the $(r+1)^{\text{th}}$ tide are approximately—

$$\left.\begin{aligned} h \cos V_m &= A_m + R_n \cos (a + re) + R_l \cos (b + re), \\ h \sin V_m &= B_m + R_n \sin (a + re) - R_l \sin (b + re) \end{aligned}\right\} \ldots\ldots (9).$$

If we take the mean of $n+1$ successive tides, the two latter terms on the right of (9) will be multiplied by $\dfrac{\sin \frac{1}{2}(n+1)e}{(n+1) \sin \frac{1}{2}e}$, and the r in the arguments $a+re$, $b+re$, will be equal to $\frac{1}{2}n$. If the $(n+2)^{\text{th}}$ tide falls exactly a semi-lunar-anomalistic period later than the first, $(n+1)e = \pi$. On account of the incommensurability of the angular velocity $\sigma - \varpi$ this condition cannot be rigorously satisfied, but if the whole series of observations be broken up into such semi-periods, then on the average of many such summations it may be taken as true.

Then, since $\frac{1}{2}e$ is a small angle,

$$(n+1) \sin \tfrac{1}{2}e = \tfrac{1}{2}\pi, \text{ and } \sin \tfrac{1}{2}(n+1)e = 1;$$

hence the factor is $2/\pi$.

Again $\frac{1}{2}ne = \frac{1}{2}\pi - \frac{1}{2}e$; thus, if $n+1$ is the mean number of tides in a semi-anomalistic period, our mean equations are—

$$\left.\begin{aligned} \frac{\pi}{2(n+1)}\{\Sigma h \cos V_m - A_m\} &= -R_n \sin (a - \tfrac{1}{2}e) - R_l \sin (b - \tfrac{1}{2}e), \\ \frac{\pi}{2(n+1)}\{\Sigma h \sin V_m - B_m\} &= R_n \cos (a - \tfrac{1}{2}e) - R_l \cos (b - \tfrac{1}{2}e), \end{aligned}\right\} (10).$$

where the summations Σ are carried out over the first semi-lunar-anomalistic period, which may be designated as 1.

In applying these equations to the next semi-period 2, the result is got by writing $a+(n+1)e$ or $a+\pi$ for a, and $b+\pi$ for b.

Thus the equations are simply the same as (10), with the signs on the *left* changed.

The equations for semi-periods 3, 4, &c., will be all identical on the right, with alternately + and − signs on the left.

Let the observations run over m semi-lunar-anomalistic periods; then double the equations appertaining to periods 2, 3, . . . $(m-1)$, and add all the m equations together, and divide by $2(m-1)$, and we have—

$$\left.\begin{aligned}\frac{\pi}{4(n+1)(m-1)}\Sigma h\cos V_m &= -R_n\sin(a-\tfrac{1}{2}e)-R_l\sin(b-\tfrac{1}{2}e),\\ \frac{\pi}{4(n+1)(m-1)}\Sigma h\sin V_m &= \quad R_n\cos(a-\tfrac{1}{2}e)-R_l\cos(b-\tfrac{1}{2}e),\end{aligned}\right\}\ldots(11),$$

where Σ now denotes summation of the following kind:—

$$\{\Sigma(1)-\Sigma(2)\}+\{\Sigma(3)-\Sigma(2)\}+\{\Sigma(3)-\Sigma(4)\}+\{\Sigma(5)-\Sigma(4)\}+\&c.,$$

the numbers (1), (2), &c., indicating the number of the semi-lunar-anomalistic-periods over which the partial sums are taken.

Suppose the whole series of observations to be reduced covers $2m+1$ *quarter*-lunar-anomalistic periods, which we denote by i, ii, iii, &c.

First suppose that the semi-period denoted previously by 1 consists of i+ii, that 2 consists of iii+iv, and so on.

Let t_o be the time of the first tide of the series, and since we take noon of the first day as epoch, t_o cannot be more than a few hours.

Let $\quad j=\frac{1}{2}e-(\sigma-\varpi)t_o=1^\circ{\cdot}6903-(\sigma-\varpi)t_o$, a small angle.

Then

$$\left.\begin{aligned}a-\tfrac{1}{2}e &= (\sigma-\varpi)t_o+\zeta_n-\tfrac{1}{2}e = \quad\zeta_n-j,\\ b-\tfrac{1}{2}e &= (\sigma-\varpi)t_o-\zeta_l-\tfrac{1}{2}e = -(\zeta_l+j).\end{aligned}\right\}\ldots\ldots(12).$$

Then denoting the operation $\frac{\pi}{4(n+1)(m-1)}\Sigma$ by S° (the mark $^\circ$ indicating that the first tide included is nearly at epoch, when $(\sigma-\varpi)t=0$), we have from (11) and (12)

$$\left.\begin{aligned}S^\circ h\cos V_m &= -R_n\sin(\zeta_n-j)+R_l\sin(\zeta_l+j),\\ S^\circ h\sin V_m &= \quad R_n\cos(\zeta_n-j)-R_l\cos(\zeta_l+j).\end{aligned}\right\}\ldots\ldots(13).$$

Secondly, suppose the semi-lunar-anomalistic period indicated by 1 consists of ii + iii, that 2 consists of iv + v, and so on.

Obviously the result is got by writing $t_0 + \frac{1}{2}\pi/(\sigma - \varpi)$ for t_0, or, what amounts to the same thing, by putting $j - \frac{1}{2}\pi$ in place of j; but we must also write $S^{\frac{1}{2}\pi}$ for S°, so as to show that the summation begins when $(\sigma - \varpi)t$ is nearly equal to $\frac{1}{2}\pi$. Then—

$$\left.\begin{aligned} S^{\frac{1}{2}\pi} h \cos V_m &= -R_n \cos(\zeta_n - j) - R_l \cos(\zeta_l + j), \\ S^{\frac{1}{2}\pi} h \sin V_m &= -R_n \sin(\zeta_n - j) - R_l \sin(\zeta_l + j). \end{aligned}\right\} \quad \ldots\ldots (14).$$

Hence

$$\left.\begin{aligned} R_n \sin(\zeta_n - j) &= -S^\circ h \cos V_m - S^{\frac{1}{2}\pi} h \sin V_m, \\ R_n \cos(\zeta_n - j) &= S^\circ h \sin V_m - S^{\frac{1}{2}\pi} h \cos V_m, \\ R_l \sin(\zeta_l + j) &= S^\circ h \cos V_m - S^{\frac{1}{2}\pi} h \sin V_m, \\ R_l \cos(\zeta_l + j) &= -S^\circ h \sin V_m - S^{\frac{1}{2}\pi} h \cos V_m. \end{aligned}\right\} \quad \ldots\ldots (15).$$

These four equations give the four unknowns R_n, ζ_n, R_l, ζ_l, and j is equal to $1^\circ{\cdot}69 - (\sigma - \varpi)t_0$.

Then if u_n, u_l denote the equilibrium arguments of the tides N and L at epoch, we have—

$$u_n = 2(h_0 - \nu) - 2(s_0 - \xi) - (s_0 - p_0),$$

$$u_l = 2(h_0 - \nu) - 2(s_0 - \xi) + (s_0 - p_0) + \pi,$$

where h_0, s_0, p_0 are the mean longitudes of moon, sun, and lunar perigee at epoch, and ν and ξ are small angles, functions of the longitude of the moon's node (tabulated in Baird's Manual).

Then if f_m is the factor of reduction (also tabulated by Baird) for the tides M_2, N, L,

$$\kappa_n = \zeta_n + u_n, \qquad \kappa_l = \zeta_l + u_l,$$

$$H_n = \frac{R_n}{f_m}, \qquad H_l = \frac{R_l}{f_m}.$$

In this investigation the interferences of the solar and diurnal tides are neglected, on the assumption that they are completely eliminated.

The difference between a lunar period and an anomalistic period is so small that the elimination of the diurnal tides will be satisfactory, but the effect of the solar tide will probably be sensible, unless we have under reduction 13 quarter-lunar-anomalistic periods, which only exceed 6 semi-lunations by about 25 hours.

The evaluation of the elliptic tides N and L from a series of observations shorter than a quarter year would be very unsatisfactory, and it is not likely that such an evaluation will be attempted. But if such a case is undertaken, the solar disturbance may be found by a plan strictly analogous to that pursued below in the case of the tides K_1, O, P. The reader may be left to deduce the requisite formulæ from the theory in § 3.

In the case of a long series of observations, each quarter year should be reduced independently, and the mean values of $H_n \cos \kappa_n$ and $H_n \sin \kappa_n$ should be adopted as the values of the functions; whence H_n and κ_n are easily found. The L tide is, of course, to be treated similarly.

§ 5. *The Tide* M_2.

This is the principal lunar tide.

If we take the mean of $n+1$ successive tides, the equations (9) give us approximately—

$$\frac{1}{n+1}\Sigma h \cos V_m = A_m, \qquad \frac{1}{n+1}\Sigma h \sin V_n = B_m \ldots . (16).$$

We here assume that in taking this mean over an exact number of semi-lunations, the lunar elliptic tides, the solar tides, and the diurnal tides are eliminated.

With respect to the elliptic tides, this condition can only be approximately satisfied, because no small number of semi-lunations is equal to a number of anomalistic periods, and the like is true of the diurnal tides. In the example given below the diurnal tides are much larger than the elliptic tides, and I have found by actual computation (the details of which are not, however, given) that the disturbance in the value of the M_2 tide arising from the diurnal tides is quite insensible, and it may be safely accepted that the same is true of the disturbance from the elliptic tides.

With respect to the disturbance arising from the principal solar tide S_2, I find that it is adequately, although not completely, eliminated by making the number $n+1$ of tides under summation Σ cover an exact number of semi-lunations.

If the whole series of observations be short, it would be pedantic to attempt a close accuracy in results, and we may accept these formulæ; if the series be long, the residual errors will be gradually completely eliminated.

We have then—

$$R_m \cos \zeta_m = A_m, \qquad R_m \sin \zeta_m = B_m.$$

If u_m be the equilibrium argument at epoch, we have

$$u_m = 2(h_o - \nu) - 2(s_o - \xi).$$

Whence
$$\kappa_m = \zeta_m + u_m, \text{ and } H_m = \frac{R_m}{f_m}.$$

The meanings of h_o, s_o, ν, ξ, f_m, have been explained in the last section.

§ 6. *The Tides* S_2 *and* K_2.

These are the principal solar and luni-solar semi-diurnal tides.

If the tide S_2 is in the same phase as K_2 at any time, three months later they are in opposite phases. Hence, for a short series of observations, the two tides cannot be separated, and both must be considered together. It is proposed to treat a long series of observations as made up of a succession of short series; hence I begin with a short series.

For the sake of brevity all the tides excepting S_2 and K_2 are omitted from the analytical expressions.

Since $$V'' = V_s + 2\eta t,$$

$$h = R_s \cos(V_s - \zeta_s) + R'' \cos(V_s + 2\eta t - \zeta''),$$

$$= \cos V_s\{R_s \cos \zeta_s + R'' \cos(2\eta t - \zeta'')\}$$

$$+ \sin V_s\{R_s \sin \zeta_s - R'' \sin(2\eta t - \zeta'')\}.$$

Hence, taking into account the equation which expresses that h is a maximum or minimum, and neglecting the variation of $2h$ or $2\eta t$ compared with that of V_s, we have—

$$h \cos V_s = R_s \cos \zeta_s + R'' \cos(2\eta t - \zeta''),$$

$$h \sin V_s = R_s \sin \zeta_s - R'' \sin(2\eta t - \zeta'').$$

The mean interval between each tide and the next is $6^{h}{\cdot}210$. Then if g be the increment of $2h$ in that period (so that with 2η equal to $0^{\circ}{\cdot}082$ per hour, g is equal to $0^{\circ}{\cdot}510$), the equations corresponding to the $(r+1)^{\text{th}}$ tide are approximately—

$$\left.\begin{aligned} h \cos V_s &= R_s \cos \zeta_s + R'' \cos(rg - \zeta''), \\ h \sin V_s &= R_s \sin \zeta_s - R'' \sin(rg - \zeta'') \end{aligned}\right\} \quad \ldots\ldots\ldots (17).$$

Now, if P be the cube of the ratio of the sun's parallax to its mean parallax, the expression for S_2, together with its parallactic inequality (the tides T, R of harmonic notation), is $PH_s \cos(2t - \kappa_s)$.

Since t is the mean solar hour angle, 2t is the same thing as V_s.

Hence $$R_s = PH_s, \quad \zeta_s = \kappa_s.$$

Also if P_o be the value of P at epoch, then for a period of two or three months we may take $P = P_o(1 + pt)$, where $P_o p$ is equal to dP/dt.

Again, if we put $\gamma = \frac{H''}{H_s}$, we have

$$R'' = f''H'' = f''\gamma H_s.$$

Also since the argument of the K_2 tide is $2t + 2h - 2\nu'' - \kappa''$, where $2\nu''$ is a certain function of the longitude of the moon's node (tabulated by Baird), and since $t = 0$, $h = h_0$ at epoch, it follows that

$$-\zeta'' = 2h_0 - 2\nu'' - \kappa''.$$

Now, when the means of the equations (17) are taken for $n+1$ successive tides, the latter terms become $\frac{\lambda_n}{\gamma} R'' \frac{\cos}{\sin}(\frac{1}{2}ng - \zeta'')$, where

$$\lambda_n = \gamma \cdot \frac{\sin \frac{1}{2}(n+1)g}{(n+1)\sin\frac{1}{2}g} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

Also, if we write

$$\left.\begin{aligned} \varpi &= 2h_0 - 2\nu'' + \tfrac{1}{2}ng, \\ \Pi &= P_0(1 + \tfrac{1}{2}np \times 6^{\text{h}}\cdot 21), \\ A_s &= \frac{1}{n+1}\Sigma h \cos V_s, \\ B_s &= \frac{1}{n+1}\Sigma h \sin V_s, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (19),$$

our equations become—

$$\left.\begin{aligned} A_s &= \Pi H_s \cos \kappa_s + f''\lambda_n H_s \cos(\varpi - \kappa''), \\ B_s &= \Pi H_s \sin \kappa_s - f''\lambda_n H_s \sin(\varpi - \kappa''). \end{aligned}\right\} \ldots\ldots\ldots (20).$$

It may be observed that Π is the mean value of P during the interval embraced by the $n+1$ tides.

In reducing a short series of observations we have to assume what is usually nearly true, viz., that $\kappa'' = \kappa_s$ and $\gamma = 0{\cdot}272$, as would be the case in the equilibrium theory of tides.

With this hypothesis, put

$$U \cos \phi = \Pi + \lambda_n f'' \cos \varpi,$$
$$U \sin \phi = \qquad \lambda_n f'' \sin \varpi,$$

from which to find U and ϕ. Then

$$A_s = H_s U \cos(\kappa_s - \phi),$$
$$B_s = H_s U \sin(\kappa_s - \phi),$$

from which to find H_s and κ_s.

Lastly, $\qquad \kappa'' = \kappa_s, \quad H'' = \gamma H_s = 0{\cdot}272\ H_s.$

In order to minimise the disturbance due to the lunar tide M_2, we have to make the $n+1$ tides cover an exact number of semi-lunations, namely, the same period as that involved in the evaluation of M_2. The elimination of the M_2 tide is adequate, although not so complete as the elimination of the effect of the S_2 tide on M_2, because M_2 is nearly three times as large as S_2.

A Long Series of Observations.—Suppose that there is a half year of observations, or two periods of six semi-lunations, each of which periods contains exactly the same number of tides.

Then each of these periods is to be reduced independently with the assumption that $\gamma = 0{\cdot}272$ and $\kappa_s = \kappa''$. If this assumption is found subsequently to be very incorrect, it might be necessary to amend these reductions by multiplying λ_n by $H'' \div 0{\cdot}272 H_s$, and by adding $\kappa_s - \kappa''$ to ω; but such repetition will not usually be necessary. From these reductions we get independent values of $H_s \cos \kappa_s$, $H_s \sin \kappa_s$ from each quarter year, and the mean of these is to be adopted, from which to compute H_s and κ_s. It remains to evaluate H'' and κ''.

The factor f'' and the angle $2\nu''$ vary so slowly that the change may be neglected from one quarter to the next, although each quarter is supposed to have been reduced with its proper values.

Let h_o and h'_o be the sun's mean longitude at the two epochs; they will clearly differ by nearly 90°, and we put $2h'_o = 2h_o + \pi + 2\delta h$. Hence it is clear that the value of ω in the second quarter is $\omega + 2\delta h + \pi$.

Thus the four equations, such as (20), appertaining to the two quarters, may be written—

$$\left.\begin{aligned} A_s &= \Pi H_s \cos \kappa_s + \frac{\lambda_n}{\gamma} \cdot f''H'' \cos(\omega - \kappa''), \\ B_s &= \Pi H_s \sin \kappa_s - \frac{\lambda_n}{\gamma} \cdot f''H'' \sin(\omega - \kappa''), \\ A_s' &= \Pi' H_s \cos \kappa_s - \frac{\lambda_n}{\gamma} \cdot f''H'' \cos(\omega + 2\delta h - \kappa''), \\ B_s' &= \Pi' H_s \sin \kappa_s + \frac{\lambda_n}{\gamma} \cdot f''H'' \sin(\omega + 2\delta h - \kappa''), \end{aligned}\right\} \quad \ldots\ldots (21),$$

where the accented symbols apply to the second quarter, and where $\frac{\lambda_n}{\gamma} = \frac{\sin \frac{1}{2}(n+1)g}{(n+1)\sin \frac{1}{2}g} = 0{\cdot}656$, a constant.

From (21),

$$A_s - A_s' - (\Pi - \Pi')H_s \cos \kappa_s = 2\frac{\lambda_n}{\gamma} \cdot f''H'' \cos \delta h \cos(\omega + \delta h - \kappa''),$$

$$-B_s + B_s' + (\Pi - \Pi')H_s \sin \kappa_s = 2\frac{\lambda_n}{\gamma} \cdot f''H'' \cos \delta h \sin(\omega + \delta h - \kappa'').$$

From these two equations, H'' and κ'' may be computed, and since $\Pi - \Pi'$ is very small, approximate values of $H_s \cos \kappa_s$, $H_s \sin \kappa_s$ suffice.

§ 7. *The Diurnal Tides* K_1, O, P.

Amongst the diurnal tides I shall only consider K_1 the luni-solar diurnal, O the principal lunar diurnal, and P the principal solar diurnal tides.

There is the same difficulty in separating P from K_1 as in the case of K_2 and S_2, and therefore in a short series of observations P and K_1 have to be treated together. It is proposed to treat a long series of observations as made up of a succession of short series; hence I begin with a short series.

For the sake of brevity all the tides excepting K_1, O, P are omitted from the analytical expressions.

If $\frac{1}{2}V_m$ denotes $(\gamma - \sigma)t$, we have

$$V' = \tfrac{1}{2}V_m + \sigma t,\ \ V_o = \tfrac{1}{2}V_m - \sigma t,\ \ V_p = \tfrac{1}{2}V_m + (\sigma - 2\eta)t, \text{ and}$$

$$h = R' \cos(\tfrac{1}{2}V_m + \sigma t - \zeta') + R_o \cos(\tfrac{1}{2}V_m - \sigma t - \zeta_o) + R_p \cos(\tfrac{1}{2}V_m + (\sigma - 2\eta)t - \zeta_p),$$

$$= \cos\tfrac{1}{2}V_m\{R' \cos(\sigma t - \zeta') + R_o \cos(\sigma t + \zeta_o) + R_p \cos((\sigma - 2\eta)t - \zeta_p)\}$$

$$+ \sin\tfrac{1}{2}V_m\{-R' \sin(\sigma t - \zeta') + R_o \sin(\sigma t + \zeta_o) - R_p \sin((\sigma - 2\eta)t - \zeta_p)\}.$$

Hence, taking account of the equation which expresses that h is a maximum or minimum, and neglecting the variation of σt compared with that of $\frac{1}{2}V_m$,* we have—

$$h \cos\tfrac{1}{2}V_m = R' \cos(\sigma t - \zeta') + R_o \cos(\sigma t + \zeta_o) + R_p \cos((\sigma - 2\eta)t - \zeta_p).$$

$$h \sin\tfrac{1}{2}V_m = -R' \sin(\sigma t - \zeta') + R_o \sin(\sigma t + \zeta_o) - R_p \sin((\sigma - 2\eta)t - \zeta_p).$$

The mean interval between each tide and the next is $6^{h}\cdot 210$.

Then if e be the increment of s, and z the increment of $s - 2h$ in that period (so that with σ equal to $0^\circ\cdot 5490$ per hour and $\sigma - 2\eta$ equal to $0^\circ\cdot 4669$ per hour, e is equal to $3^\circ\cdot 4095$ and z equal to $2^\circ\cdot 8994$); and if a, b, c denote the values of $\sigma t - \zeta'$, $\sigma t + \zeta_o$, $(\sigma - 2\eta)t - \zeta_p$ at the time of the first tide under consideration, the equations corresponding to the $(r+1)^{\text{th}}$ tide are approximately—

$$\left.\begin{aligned} h \cos\tfrac{1}{2}V_m &= R' \cos(a + re) + R_o \cos(b + re) + R_p \cos(c + rz), \\ h \sin\tfrac{1}{2}V_m &= -R' \sin(a + re) + R_o \sin(b + re) - R_p \sin(c + rz). \end{aligned}\right\} \quad (22).$$

If we take the mean of $n+1$ successive tides, the first pair of terms

* I have satisfied myself by analysis, which I do not reproduce, that on taking means this error becomes very small.

will be multiplied by $\frac{\sin \frac{1}{2}(n+1)e}{(n+1)\sin \frac{1}{2}e}$ and the last term by the similar function with z in place of e; also the r in the arguments must be put equal to $\frac{1}{2}n$.

If the $(n+2)^{\text{th}}$ tide falls exactly a semi-lunar period later than the first, $(n+1)e = \pi$. On account of the incommensurability of the angular velocity σ, this condition cannot be rigorously satisfied, but if the whole series of observations be broken up into such semi-periods, then, on the average of many such summations, it may be taken as true.

Since $\frac{1}{2}e$ is a small angle, $(n+1)\sin \frac{1}{2}e = \frac{1}{2}\pi$, and $\sin \frac{1}{2}(n+1)e = 1$; hence the first factor is equal to $2/\pi$.

Again,

$$\tfrac{1}{2}(n+1)z = \tfrac{1}{2}(n+1)e \,.\, \frac{z}{e} = \tfrac{1}{2}\pi \,.\, \frac{\sigma-2\eta}{\sigma} = 76^\circ\ 32' \text{ in degrees};$$

and $(n+1)\sin \frac{1}{2}z = \frac{1}{2}\pi \,.\, \frac{\sigma-2\eta}{\sigma}$.

Therefore

$$\frac{\sin \frac{1}{2}(n+1)z}{(n+1)\sin \frac{1}{2}z} = \frac{2}{\pi}\cdot\frac{\sigma}{\sigma-2\eta}\sin 76^\circ\ 32' = \frac{2}{\pi}\times 1{\cdot}1436 = \frac{2}{\pi}\times\lambda, \text{ suppose.}$$

Again $\quad \frac{1}{2}ne = \frac{1}{2}\pi - \frac{1}{2}e = \frac{1}{2}\pi - 1^\circ{\cdot}7048,$

$$\tfrac{1}{2}nz = \tfrac{1}{2}\pi - 13^\circ{\cdot}4647 - 1^\circ{\cdot}4497 = \tfrac{1}{2}\pi - 14^\circ{\cdot}9144.$$

Now let

$$\left.\begin{aligned} \alpha &= a - 1^\circ{\cdot}7048, \\ \beta &= b - 1^\circ{\cdot}7048, \\ \gamma &= c - 14^\circ{\cdot}9144, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23),$$

and we have

$$\left.\begin{aligned} a + \tfrac{1}{2}ne &= \tfrac{1}{2}\pi + \alpha, \\ b + \tfrac{1}{2}ne &= \tfrac{1}{2}\pi + \beta, \\ c + \tfrac{1}{2}nz &= \tfrac{1}{2}\pi + \gamma. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24).$$

Thus, if $n+1$ is the mean number of tides in a semi-lunar period, the means of equations (22) become

$$\left.\begin{aligned} \frac{\pi}{2(n+1)}\Sigma h \cos \tfrac{1}{2}V_m &= -R' \sin \alpha - R_o \sin \beta - \lambda R_p \sin \gamma, \\ \frac{\pi}{2(n+1)}\Sigma h \sin \tfrac{1}{2}V_m &= -R' \cos \alpha + R_o \cos \beta - \lambda R_p \cos \gamma, \end{aligned}\right\} \ldots \quad (25),$$

where the summations are carried out over the first semi-lunar period, which may be designated as 1.

In applying these equations to the next semi-period 2, the result is obtained by writing $a+(n+1)e$ for a, $b+(n+1)e$ for b, and $c+(n+1)z$ for c; that is to say, $a+\pi$ for a, $b+\pi$ for b, and $c+153°\cdot0706$ or $c+\pi-26°\cdot9294$ for c.

If, therefore, we put $\epsilon = 26°\cdot9294$, we obtain the result from (25) by changing the signs on the left and writing $\gamma-\epsilon$ for γ.

The equations for semi-periods 3, 4, 5, &c., will be alternately + and − on the left, and identical as regards the terms in α and β, but with $\gamma-2\epsilon$, $\gamma-3\epsilon$, $\gamma-4\epsilon$, &c., successively in place of γ.

Let the observations run over m semi-lunar periods; then double the equations appertaining to periods 2, 3 . . . $(m-1)$, add all the m equations together, and divide by $2(m-1)$.

The terms in R_p will involve the series

$$\frac{\sin}{\cos}\gamma+2\frac{\sin}{\cos}(\gamma-\epsilon)+2\frac{\sin}{\cos}(\gamma-2\epsilon)+\ldots+\frac{\sin}{\cos}(\gamma-(m-1)\epsilon).$$

This is equal to

$$2\frac{\sin\frac{1}{2}(m-1)\epsilon}{\tan\frac{1}{2}\epsilon}\frac{\sin}{\cos}(\gamma-\tfrac{1}{2}(m-1)\epsilon).$$

Then if we put

$$\mu=\frac{\lambda\sin\frac{1}{2}(m-1)\epsilon}{(m-1)\tan\frac{1}{2}\epsilon},\quad\text{where } \lambda = 1\cdot1436,$$

our equations (25) become

$$\left.\begin{aligned}&\frac{\pi}{4(n+1)(m-1)}\Sigma h\cos\tfrac{1}{2}V_m\\&\qquad=-R'\sin\alpha-R_0\sin\beta-\mu R_p\sin(\gamma-\tfrac{1}{2}(m-1)\epsilon),\\&\frac{\pi}{4(n+1)(m-1)}\Sigma h\sin\tfrac{1}{2}V_m\\&\qquad=-R'\cos\alpha+R_0\cos\beta-\mu R_p\cos(\gamma-\tfrac{1}{2}(m-1)\epsilon),\end{aligned}\right\}\ldots\ (26),$$

where Σ now denotes summation of the following kind:—

$$\{\Sigma(1)-\Sigma(2)\}+\{\Sigma(3)-\Sigma(2)\}+\{\Sigma(3)-\Sigma(4)\}+\ldots$$

Suppose the whole series of observations to be reduced covers exactly $2m+1$ *quarter*-lunar periods, which we denote by I, II, III, &c.

First suppose that the semi-period denoted previously by 1 consists of I+II, that 2 consists of III+IV, and so on.

Let t_0 denote the time of the first tide of the series, and since noon of the first day is epoch, t_0 cannot be more than a few hours.

$$\left.\begin{aligned}&\text{Let}\quad i=\tfrac{1}{2}e-\sigma t_0=1°\cdot7048-\sigma t_0,\\&\text{and}\quad k=\tfrac{1}{2}z-(\sigma-2\eta)t_0=1°\cdot4497-(\sigma-2\eta)t_0;\end{aligned}\right\}\ldots\ (27),$$

i and k are clearly small angles.

Since $\epsilon = 26^\circ{\cdot}9294$, $\frac{1}{2}\epsilon + 1^\circ{\cdot}4497 = 14^\circ{\cdot}9144$; and from (23)

$$\left.\begin{aligned} \alpha &= \sigma t_0 - \zeta' - 1^\circ{\cdot}7048 = -(\zeta' + i), \\ \beta &= \sigma t_0 + \zeta_0 - 1^\circ{\cdot}7048 = \quad (\zeta_0 - i), \\ \gamma &= (\sigma - 2\eta) t_0 - \zeta_p - 14^\circ{\cdot}9144 = -\zeta_p - k - \tfrac{1}{2}\epsilon \end{aligned}\right\} \ldots\ldots (28).$$

If the same notation be adopted as that explained in § 4 (the only difference being that we now deal with quarter-lunar instead of quarter-anomalistic periods), we have

$$\begin{aligned} S^\circ h \cos \tfrac{1}{2} V_m &= R' \sin(\zeta' + i) - R_0 \sin(\zeta_0 - i) + \mu R_p \sin(\zeta_p + k + \tfrac{1}{2} m\epsilon), \\ S^\circ h \sin \tfrac{1}{2} V_m &= -R' \cos(\zeta' + i) + R_0 \cos(\zeta_0 - i) - \mu R_p \cos(\zeta_p + k + \tfrac{1}{2} m\epsilon), \end{aligned} \quad \ldots (29).$$

Secondly, suppose the semi-lunar period indicated by 1 consists of II + III, that 2 consists of IV + V, and so on. Then, obviously, the result is got by writing $t_0 + \frac{1}{2}\pi/\sigma$ for t_0; that is to say, write $i - \frac{1}{2}\pi$ for i and $k - \frac{1}{2}(\sigma - 2\eta)\pi/\sigma$, or $k - \frac{1}{2}\pi + \eta\pi/\sigma$ for k. But $\eta\pi/\sigma$ is equal to $\frac{1}{2}\epsilon$, and we write $k - \frac{1}{2}\pi + \frac{1}{2}\epsilon$ for k. Therefore, following the notation used in §4 for N and L,

$$\left.\begin{aligned} S^{\frac{1}{2}\pi} h \cos \tfrac{1}{2} V_m &= -R' \cos(\zeta' + i) - R_0 \cos(\zeta_0 - i) \\ &\qquad - \mu R_p \cos(\zeta_p + k + \tfrac{1}{2}(m+1)\epsilon), \\ S^{\frac{1}{2}\pi} h \sin \tfrac{1}{2} V_m &= -R' \sin(\zeta' + i) - R_0 \sin(\zeta_0 - i) \\ &\qquad - \mu R_p \sin(\zeta_p + k + \tfrac{1}{2}(m+1)\epsilon). \end{aligned}\right\} (30).$$

These four S's require correction for the disturbance due to the semi-diurnal terms M_2 and S_2, and I shall return to this point later. In the meantime write

$$\left.\begin{matrix} W \\ X \end{matrix}\right\} = S^\circ h \begin{matrix}\cos \\ \sin\end{matrix} \tfrac{1}{2} V_m + \text{corr.}, \qquad \left.\begin{matrix} Y \\ Z \end{matrix}\right\} = S^{\frac{1}{2}\pi} h \begin{matrix}\cos \\ \sin\end{matrix} \tfrac{1}{2} V_m + \text{corr.} \quad (31),$$

and we have—

$$\left.\begin{aligned} \tfrac{1}{2}(W + Z) &= -R_0 \sin(\zeta_0 - i) - \mu R_p \sin \tfrac{1}{4}\epsilon \cos(\zeta_p + k + \tfrac{1}{4}(2m+1)\epsilon), \\ \tfrac{1}{2}(X - Y) &= \quad R_0 \cos(\zeta_0 - i) - \mu R_p \sin \tfrac{1}{4}\epsilon \sin(\zeta_p + k + \tfrac{1}{4}(2m+1)\epsilon), \end{aligned}\right\} (32),$$

$$\left.\begin{aligned} \tfrac{1}{2}(W - Z) &= \quad R' \sin(\zeta' + i) + \mu R_p \cos \tfrac{1}{4}\epsilon \sin(\zeta_p + k + \tfrac{1}{4}(2m+1)\epsilon), \\ \tfrac{1}{2}(X + Y) &= -R' \cos(\zeta' + i) - \mu R_p \cos \tfrac{1}{4}\epsilon \cos(\zeta_p + k + \tfrac{1}{4}(2m+1)\epsilon). \end{aligned}\right\} (33).$$

If we put

$$\left.\begin{aligned} L &= \{\tfrac{1}{2}(X + Y) + R' \cos(\zeta' + i)\} \tan \tfrac{1}{4}\epsilon, \\ M &= \{\tfrac{1}{2}(W - Z) - R' \sin(\zeta' + i)\} \tan \tfrac{1}{4}\epsilon, \end{aligned}\right\} \ldots\ldots (34),$$

the equations (33) may be written—

$$\left.\begin{aligned}\tfrac{1}{2}(W+Z)-L &= -R_o \sin(\zeta_o - i),\\ \tfrac{1}{2}(X-Y)+M &= \ \ R_o \cos(\zeta_o - i).\end{aligned}\right\}\quad\ldots\ldots\ldots (35).$$

The four equations (32), (33) involve six unknown quantities, R', ζ', R_o, ζ_o, R_p, ζ_p, and are insufficient for their determination.

In reducing a short series of observations it is necessary to assume what is usually nearly true, viz., that $\kappa_p = \kappa'$, and $H_p/H' = 0{\cdot}3309$, as would be the case in the equilibrium theory of tides.

Then, writing q for 0·3309, we have approximately $R_p = H_p = qH'$. The argument of the K_1 tide is $t + (h - \nu') - \frac{1}{2}\pi - \kappa'$, where ν' is a certain function of the longitude of the moon's node (tabulated by Baird); and the argument of the P tide is $t - h + \frac{1}{2}\pi - \kappa_p$.

At the noon which is taken as epoch $t = 0$, $h = h_o$, and the two arguments are equal to $-\zeta'$ and $-\zeta_p$.

Hence
$$-\zeta' = \ \ h_o - \nu' - \tfrac{1}{2}\pi - \kappa',$$
$$-\zeta_p = -h_o + \tfrac{1}{2}\pi - \kappa_p.$$

Therefore
$$\zeta_p = \zeta' + 2h_o - \nu' - \pi + (\kappa_p - \kappa').$$

Putting $\kappa_p = \kappa'$ as explained above,

$$\zeta_p + k = \zeta' + i + 2h_o - \nu' - \pi + l,$$

where
$$l = k - i = [1^\circ{\cdot}450 - (\sigma - 2\eta)t_o] - [1^\circ{\cdot}705 - \sigma t_o],$$
$$= -0^\circ{\cdot}255 + 2\eta t_o, \text{ a small angle} \ldots\ldots\ldots\ldots (36).$$

Then, if
$$\left.\begin{aligned}\theta &= 2h_o - \nu' + l + \tfrac{1}{4}(2m+1)\epsilon,\\ \rho_m &= q\mu \cos \tfrac{1}{4}\epsilon,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots (37),$$

we have

$$\left.\begin{aligned}\tfrac{1}{2}(W-Z) &= \ \ f'H' \sin(\zeta' + i) - \rho_m H' \sin(\zeta' + i + \theta),\\ \tfrac{1}{2}(X+Y) &= -f'H' \cos(\zeta' + i) + \rho_m H' \cos(\zeta' + i + \theta).\end{aligned}\right\}\ldots (38).$$

Let
$$\left.\begin{aligned}T\cos\psi &= f' - \rho_m \cos\theta,\\ T\sin\psi &= \ \ \ \ \rho_m \sin\theta.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

whence T and ψ may be computed ; and

$$\left.\begin{aligned}\tfrac{1}{2}(W-Z) &= \ \ H'T \sin(\zeta' + i - \psi),\\ \tfrac{1}{2}(X+Y) &= -H'T \cos(\zeta' + i - \psi).\end{aligned}\right\}\quad\ldots\ldots\ldots (40).$$

From these we compute H' and ζ' and $\zeta' + i$.

Then if $u' = h_o - \nu' - \frac{1}{2}\pi$, the equilibrium argument at epoch of K_1,

$$\kappa' = \zeta' + u'.$$

We have also $H_p = qH' = 0{\cdot}3309\ H'$, $\kappa_p = \kappa'$.

Returning to equations (34) and (35), we compute $R' = f'H'$, and hence $R' \frac{\cos}{\sin} (\zeta' + i)$, and then L and M.

Having these, we compute R_o and ζ_o from (35).

Then, if $u_o = h_o - \nu - 2(s_o - \xi) + \frac{1}{2}\pi$, the equilibrium argument at epoch of O,

$$\kappa_o = \zeta_o + u_o, \text{ and } R_o = \frac{H_o}{f_o},$$

where f_o is a certain function of the longitude of the moon's node, tabulated in Baird's Manual.

A Long Series of Observations.—Suppose that there is a half year of observation, or two periods of thirteen quarter-lunar periods, each of which contains exactly the same number of tides.

Then each of these periods is to be reduced independently with the assumption that $q = 0{\cdot}3309$ and $\kappa_p = \kappa'$. If this assumption be found subsequently to be very incorrect, it might be necessary to amend these reductions by adding $\kappa_p - \kappa'$ to the value of θ, and by multiplying ρ_m by $H_p \div 0{\cdot}3309\ H'$, but such repetition will not usually be necessary.

From these reductions we get independent values of $H' \cos \kappa'$, $H' \sin \kappa'$, $H_o \cos \kappa_o$, $H_o \sin \kappa_o$ from each quarter year, and the means of these are to be adopted from which to compute H', κ', H_o, κ_o.

It remains to evaluate H_p and κ_p.

The factor f' and the angle ν' vary so slowly that the change from one quarter to the next may be neglected, although each quarter is supposed to have been reduced with its proper values.

Let h_o, h'_o be the values of the sun's mean longitude at the two epochs; then since the second epoch is nearly a quarter year later than the first, h'_o will exceed h_o by about 90°.

Let $h'_o = h_o + \frac{1}{2}\pi + \delta h$, so that δh is small.

If $\zeta' + \delta\zeta'$, $\zeta_p + \delta\zeta_p$ be the values of ζ', ζ_p at the second epoch, we have $\zeta' + \delta\zeta' = -h'_o + \nu' + \frac{1}{2}\pi + \kappa'$, $\zeta' = -h_o + \nu' + \frac{1}{2}\pi + \kappa'$, and therefore $\delta\zeta' = -\frac{1}{2}\pi - \delta h$.

Again, $\zeta_p + \delta\zeta_p = h'_o - \frac{1}{2}\pi + \kappa_p$, $\zeta_p = h_o - \frac{1}{2}\pi + \kappa_p$, and therefore $\delta\zeta_p = \frac{1}{2}\pi + \delta h$.

Let $i + \delta i$, $k + \delta k$ be the values of i and k corresponding to the second epoch, and let W′, X′, Y′, Z′ be the values of those quantities in the second quarter. Then, replacing $\frac{1}{4}(2m+1)\epsilon$ by 87°·5, since that is its value when $2m+1$ is 13, we have from (33)

$$\tfrac{1}{2}(W'-Z') = -R'\cos(\zeta'+i+\delta i-\delta h) + \mu R_p \cos\tfrac{1}{4}\epsilon\cos(\zeta_p+k+\delta k+\delta h+87^\circ{\cdot}5),$$

$$\tfrac{1}{2}(X'+Y') = -R'\sin(\zeta'+i+\delta i-\delta h) + \mu R_p \cos\tfrac{1}{4}\epsilon\sin(\zeta_p+k+\delta k+\delta h+87^\circ{\cdot}5),$$

$$\tfrac{1}{2}(W-Z) = R'\sin(\zeta'+i) + \mu R_p\cos\tfrac{1}{4}\epsilon\sin(\zeta_p+k+87^\circ{\cdot}5),$$

$$\tfrac{1}{2}(X+Y) = -R'\cos(\zeta'+i) - \mu R_p\cos\tfrac{1}{4}\epsilon\cos(\zeta_p+k+87^\circ{\cdot}5).$$

Hence

$$\left.\begin{aligned} \tfrac{1}{4}(W'-Z')-\tfrac{1}{4}(X+Y) &= R'\sin\tfrac{1}{2}(\delta i-\delta h)\sin(\zeta'+i+\tfrac{1}{2}(\delta i-\delta h)) \\ &\quad + \mu R_p\cos\tfrac{1}{4}\epsilon\cos\tfrac{1}{2}(\delta k+\delta h)\cos(\zeta_p+k+\tfrac{1}{2}(\delta k+\delta h)+87^\circ{\cdot}5), \\ \tfrac{1}{4}(W-Z)+\tfrac{1}{4}(X'+Y') &= -R'\sin\tfrac{1}{2}(\delta i-\delta h)\cos(\zeta'+i+\tfrac{1}{2}(\delta i-\delta h)) \\ &\quad + \mu R_p\cos\tfrac{1}{4}\epsilon\cos\tfrac{1}{2}(\delta k+\delta h)\sin(\zeta_p+k+\tfrac{1}{2}(\delta k+\delta h)+87^\circ{\cdot}5). \end{aligned}\right\} \quad (41).$$

In these equations R' is equal to f$'$H$'$ and R_p is equal to H_p.

The terms involving R' are clearly small, and approximate values of R' and ζ', as derived from the first quarter, will be sufficient to compute them. Afterwards we can compute R_p or H_p and ζ_p; then if u_p denotes $-h_o+\frac{1}{2}\pi$, the equilibrium argument of P at the first epoch, $\kappa_p = \zeta_p+u_p$.

The values of H_p, κ_p thus deduced ought not to differ very largely from those assumed in the two independent reductions.

The same investigation serves for the evaluation of the P tide from any two sets of observations, each consisting of thirteen quarter-lunar periods, and with a small change in the analysis we need not suppose each to consist of thirteen such periods. But the two epochs must be such that sin δh is small and cos δh is large, or the formulæ, although analytically correct, will fail in their object.

§ 8.—*The Disturbance of* K_1, O, P *due to* M_2 *and* S_2.

It has been remarked in § 7 that the diurnal tides are perturbed by the semi-diurnal. The general method has been given in § 3, by which to calculate the effect on any one tide, whose increment of argument since epoch is V_p and speed is p, due to a tide whose increment is V_q and speed q.

Since in the present instance all the diurnal tides have been consolidated into one of speed $\gamma-\sigma$, we have to calculate the effect of the tides whose speeds are $2(\gamma-\sigma)$ and $2(\gamma-\eta)$ on the tide whose speed is $\gamma-\sigma$. It follows, therefore, that the factor q/p or k_q of (3) is in the first case equal to $2(\gamma-\sigma)/(\gamma-\sigma)$ or 2, and in the second case is $2(\gamma-\eta)/(\gamma-\sigma)$ or 2·070; or $k_m = 2$, $k_s = 2{\cdot}070$.

The coefficients F, G, f, g, as due to the tide M_2 of speed $2(\gamma-\sigma)$, will be written with suffix m, and as due to the tide S_2 of speed $2(\gamma-\eta)$, with suffix s. The sums and means have also to be taken in the two ways denoted by S° and $S^{\frac{1}{2}\pi}$. Hence we have altogether to compute sixteen coefficients, which by an easily intelligible notation may be written $F_m^{(o)}$, $G_m^{(o)}$, . . . $f_s^{(\frac{1}{2}\pi)}$, $g_{s(\frac{1}{2}\pi)}$.

In order to compute the sixteen coefficients, it is necessary to find the mean cosines and sines of the four following angles, viz.:—

$\frac{1}{2}V_m \pm V_m$, $\frac{1}{2}V_m \pm V_s$, and the means have to be taken in the two ways denoted S° and $S^{\frac{1}{2}\pi}$.

These means are exactly the same in form as what the means of $h\cos$ and $h\sin$ (which had to be evaluated in S° and $S^{\frac{1}{2}\pi}$) would be if all the heights were regarded as positive unity, irrespective of whether they are H.W. or L.W. Hence the same plan of computation serves here as elsewhere; the plan is explained in the following section.

By comparison of equation (7) and the definitions (31) of W, X, Y, Z in the last section, we have:—

$$\left.\begin{aligned}
W &= S^\circ h\cos\tfrac{1}{2}V_m - \{A_m F_m^{(o)} + B_m G_m^{(o)} + A_s F_s^{(o)} + B_s G_s^{(o)}\}, \\
X &= S^\circ h\sin\tfrac{1}{2}V_m - \{A_m f_m^{(o)} + B_m g_m^{(o)} + A_s f_s^{(o)} + B_s g_s^{(o)}\}, \\
Y &= S^{\frac{1}{2}\pi} h\cos\tfrac{1}{2}V_m - \{A_m F_m^{(\frac{1}{2}\pi)} + B_m G_m^{(\frac{1}{2}\pi)} + A_s F_s^{(\frac{1}{2}\pi)} + B_s G_s^{(\frac{1}{2}\pi)}\}, \\
Z &= S^{\frac{1}{2}\pi} h\sin\tfrac{1}{2}V_m - \{A_m f_m^{(\frac{1}{2}\pi)} + B_m g_m^{(\frac{1}{2}\pi)} + A_s f_s^{(\frac{1}{2}\pi)} + B_s g_s^{(\frac{1}{2}\pi)}\}.
\end{aligned}\right\} \quad (42).$$

The four quantities A_m, B_m, A_s, B_s are known from the evaluations of the tides M_2 and S_2; whence the corrections referred to in § 7 are calculable.

§ 9. *On the Summations.*

It will be seen from the preceding sections that sums have to be found of the following functions:—

$$h\frac{\cos}{\sin}V_m,\quad h\frac{\cos}{\sin}V_s,\quad h\frac{\cos}{\sin}\tfrac{1}{2}V_m;$$

and also of

$$\frac{\cos}{\sin}\tfrac{1}{2}V_m,\quad \frac{\cos}{\sin}\tfrac{3}{2}V_m,\quad \frac{\cos}{\sin}(\tfrac{1}{2}V_m \pm V_s).$$

It is necessary to calculate the five angles $\frac{1}{2}V_m$, V_s, $\frac{1}{2}V_m \pm V_s$, and V_m, for each tide, and the reader will easily see, by the example in the Appendix, how they may be computed with considerable rapidity, by aid of an auxiliary table A.

The computation of sines and cosines and multiplication by heights, may, with sufficient accuracy, be abridged, by regarding the cosine or sine of any angle lying within a given 5° of the circumference as equal to the cosine or sine of the middle of that 5°.

The process then consists in the grouping of the heights according to the values of their V's (V_m, V_s, $\frac{1}{2}V_m$, as the case may be). The heights in each group are then summed. Since the L.W. heights are all negative, they are treated in a separate table, and are considered as positive until their combination with the H.W. at a later stage. We shall, for the present, only speak of one of these groupings, taking it as a type of both.

Since $\frac{\cos}{\sin}(\alpha+180°) = -\frac{\cos}{\sin}\alpha$, the eighteen groups forming the 3rd quadrant may be thrown in with the 1st quadrant by a mere change of sign; and the like is true of the 4th and 2nd quadrants.

Since $\cos(180°-\alpha) = -\cos\alpha$ and $\sin(180°-\alpha) = \sin\alpha$, it follows that we have to go through the 2nd quadrant in reversed order, in order to fall in with the succession which holds in the 1st quadrant, and, moreover, the cosine changes its sign, whilst the sine does not do so. Hence the following schemes will give us the eighteen groups which all have the same cosines and sines:—

for cosines

$$(1^{st}-3^{rd})-(2^{nd}-4^{th}) \text{ reversed},$$

for sines

$$(1^{st}-3^{rd})+(2^{nd}-4^{th}) \text{ reversed}.$$

Thus, one grouping of the heights serves for both cosines and sines, and, save for the last step, the additions are the same.

The combination of the H.W. and L.W. results is best made at the stage where $1^{st}-3^{rd}$ and $2^{nd}-4^{th}$ have been formed.

The negative signs for the L.W. results are introduced before addition to the H.W. results, and total $1^{st}-3^{rd}$ and $2^{nd}-4^{th}$ are thus formed.

After the eighteen cosine and sine total numbers are thus formed, they are to be multiplied by the cosines or sines of 2° 30′, 7° 30′, 12° 30′, ... 87° 30′. The products are then summed so as to give $\Sigma h \frac{\cos}{\sin}$.

It was noted at the beginning of this section that we also have sums of the form $\Sigma \frac{\cos}{\sin}$. These sums are obviously made by entering unity in place of each height, and, of course, not treating the L.W. as negative. Thus, where the H.W. and L.W. are combined it is not necessary to change the sign of the L.W., as was done in the combination of H.W. and L.W. for $\Sigma h \frac{\cos}{\sin}$. These summations are considerably less laborious than the others.

In the case of the tides M_2 and S_2, the division of the sums $\Sigma h \frac{\cos}{\sin}$

by the total number of entries gives the required results. But for N, L, and similarly for the diurnal tides K_1, O, P, the grouping and summations have to be broken into a number of subordinate periods, which are to be operated on to form S° and $S^{\frac{1}{2}\pi}$. The multiplication by the eighteen mean cosines and sines is best deferred to a late stage in the computation.

Thus, for example, for N and L, the quarter-lunar-anomalistic periods, i, ii, iii, &c., are treated independently, and we find $(1^{st}-3^{rd})\pm(2^{nd}-4^{th}$ reversed) for each. There are thus eighteen cosine numbers and eighteen sine numbers for each of i, ii, iii, &c.

We next form the sums two and two, i+ii, iii+iv, &c.; next find the differences (i+ii)−(iii+iv), (v+vi)−(iii+iv), &c.; add the differences together; then multiply by the eighteen cosines or sines of $2\frac{1}{2}^\circ$, $7\frac{1}{2}^\circ$, &c., and finally multiply by $\frac{\pi}{4(n+1)(m-1)}$, and so find $S^\circ h \frac{\cos}{\sin}$.

We next go through exactly the same process, but beginning with ii instead of i, and so find $S^{\frac{1}{2}\pi} h \frac{\cos}{\sin}$.

The same process applies, *mutatis mutandis*, for finding S° and $S^{\frac{1}{2}\pi} \frac{\cos}{\sin}$.

There are two cases which merit attention in particular. The sorting of heights in quarter-lunar-anomalistic periods, according to values of V_m, serves, in the first instance, for the evaluation of N and L, but it serves, secondly, to evaluate M_2, for we then simply neglect the subdivision into quarter periods and treat the whole as one series, but stop at the end of a semi-lunation.

The sorting of heights in quarter-lunar periods, according to the values of $\frac{1}{2}V_m$, also serves several purposes.

We first find from it S° and $S^{\frac{1}{2}\pi} h \frac{\cos}{\sin} \frac{1}{2}V_m$, and secondly, by merely counting the entries in each group for each quarter period, instead of adding up the heights, we arrive at S° and $S^{\frac{1}{2}\pi} \cos \frac{1}{2}V_m$. (It may be noted in passing that what is wanted, according to preceding analysis, is the sum of $\frac{\cos}{\sin}(\frac{1}{2}V_m - V_m)$, so that there will be a change of sign in the sine sum to get the desired result.)

But, besides these, S° and $S^{\frac{1}{2}\pi} \frac{\cos}{\sin}(\frac{1}{2}V_m + V_m)$ can be obtained with sufficient accuracy from the same sorting.

The angles $\frac{1}{2}V_m$ were sorted in four times eighteen groups, for each quarter-lunar period. If each angle were multiplied by three, the eighteen entries of the 1^{st} quadrant would be converted into three groups of six, lying in three quadrants, viz., I^{st}, II^{nd}, III^{rd}; the

2^{nd} quadrant is changed to IV^{th}, I^{st}, II^{nd}; the 3^{rd} to III^{rd}, IV^{th}, I^{st}; and the 4^{th} to II^{nd}, III^{rd}, IV^{th}. Hence eighteen entries of $1^{st}-3^{rd}$ are converted into three sixes, $I^{st}-III^{rd}$, $II^{nd}-IV^{th}$, $-\{I^{st}-III^{rd}\}$; and eighteen entries of $2^{nd}-4^{th}$ are converted into three sixes, $-\{II^{nd}-IV^{th}\}$, $I^{st}-III^{rd}$, $II^{nd}-IV^{th}$.

Hence a new $I^{st}-III^{rd}$ of six entries is made up thus:—

first six of former	$1^{st}-3^{rd}$
+ second six of former	$2^{nd}-4^{th}$
− third six of former	$1^{st}-3^{rd}$.

And a new $II^{nd}-IV^{th}$ of six entries is made up of

− first six of former	$2^{nd}-4^{th}$
+ second six of former	$1^{st}-3^{rd}$
+ third six of former	$2^{nd}-4^{th}$.

These $I^{st}-III^{rd}$ and $II^{nd}-IV^{th}$ may now be treated just like the other ones. Thus, without calculating $\frac{3}{2}V_m$, we have from the former $1^{st}-3^{rd}$ and $2^{nd}-4^{th}$ the results of a fresh grouping according to values of $\frac{3}{2}V_m$.

It is true that there is a considerable loss of accuracy, because all angles within 15° are now treated as having the same sine and cosine.

§ 10. *Rules for the Partition of the Observations into Groups.*

It appears from the preceding investigations that it is required to divide up the observations into groups. This may be done, with all necessary accuracy, and with great convenience, by dividing the tides just as they would be divided if every H.W. followed L.W., and *vice versâ*, at the mean interval of $6^{h}\cdot2103$.

Now a quarter-lunar-anomalistic period is $165^{h}\cdot3272$, a quarter-lunar period is $163^{h}\cdot9295$, and semi-lunation is $354^{h}\cdot3670$. Hence, dividing these numbers by $6^{h}\cdot2103$, we find that there are 26·62145 tides in a quarter-anomalistic period, 26·3964 in a quarter period, and 57·0612 in a semi-lunation.

It may be remarked in passing that these results show that the $n+1$ of (10), § 4, is 53·243, and the $n+1$ of (25), § 7, is 52·793.

It is, of course, impossible to have a fractional number of tides, and, therefore, we make a small multiplication table of these numbers, and take the nearest integer in each case. For example, in the case of the semi-lunations, we have—

1.	57·0612	57.	4.	228·2448	228.
2.	114·1224	114.	5.	285·3060	285.
3.	171·1836	171.	6.	342·3672	342.

These have to be divided between H. and L.W. For the sake of convenience, I suppose that we always begin the series with a H.W., then when the integer is odd we put in one more H.W. than L.W., and thus have the following rule:—

No. of semi-lunation ..	1	2	3	4	5	6
No. of last H.W. in the semi-lunation	29	57	86	114	143	171
No. of last L.W. in the semi-lunation	28	57	85	114	142	171

The H.W. and L.W. are here supposed to be numbered consecutively from 1 onwards in separate tables.

The other rules of partition given in Appendix E are found in the same way.

§ 11. *On the Over Tides.*

Observations of H. and L.W. are very inappropriate for the determination of these tides (of which the most important are M_4, M_6, S_4, S_6), because they express the departure of the wave from the simple harmonic shape, and we are supposed to have no information as to what occurs between two tides. These tides make the interval from H. to L.W. longer than from L. to H.W., and there is no doubt that, assuming the existence in the expression for h of a term of the form $A_{2m}\cos 2V_m + B_{2m}\sin 2V_m$, we shall get an approximation to A_{2m} and B_{2m} by finding the mean of $h\cos 2V_m$ and $h\sin 2V_m$. But the computation of the F, G, f, g, coefficients for the perturbation of M_4 by M_2 would be essential, and thus the amount of additional computation would be very great, whereas in the analysis of continuous observation the overtides are found almost without any additional work. I am inclined to think that it would be best to obtain hourly observations for several days at several parts of a lunation, and by some methods of interpolation to construct a typical semi-diurnal tide-wave, from which, by the ordinary methods of harmonic analysis, we could find the ratio of the heights of the over-tides to the fundamental, and the relationship of their phases.

I make no attempt at such an investigation in this place.

§ 12. *On the Annual and Semi-annual Tides.*

These tides are frequently of much importance, so that they ought not to be neglected from a navigational point of view. It is obviously impossible to obtain any results from a series of observations of less than a year's duration.

Rules for the partition of tides into months or 12th parts of a year are given in the Appendix E. The mean of all the H. and L.W. observations for each month may be taken as the height of mean water at the middle of the month, and the 12 values for the year may be submitted to the ordinary processes of harmonic analysis for the evaluation of these two tides.

We have supposed in the previous investigation that the tide heights are measured from mean sea-level, and although it is not necessary that this condition should be rigorously satisfied, it might be well, where there is a large annual tide, to refer the heights to different datum levels in the different quarters of the year.

§ 13. *On Gaps in the Series of Observations.*

It often happens in actual observations that a few tides are missing through some accident, or are obviously vitiated by heavy weather. Now the present method depends for its applicability on the evanescence of terms in the averages. It is true that it is rigorously applicable even for scattered observations, but if applied to such a case all the F, G, f, g coefficients have to be calculated, and, as every tide reacts on every other, the computation would be so extensive as to make the method almost impracticable. Thus, where there is a gap, observations must be fabricated (of course noting that they are fabrications) by some sort of interpolation, and even values which are very incorrect are better than none.* If the interpolation is extensive, it might be well to test its correctness in a few places when the reduction is done. If a whole week or fortnight be missing, and if the computer cannot find a plausible method of interpolation, I can only suggest a preliminary reduction from the continuous parts, and the computation of a tide table for the hiatus. Each such case must be treated on its merits, and it is hardly possible to formulate general rules.

APPENDIX.

Tables and Rules of General Applicability.

A. *To find* $\frac{1}{2}V_m$.

The following table is for finding what would be the mean moon's hour-angle, if the moon had been on the meridian at the epoch. This angle is denoted by $\frac{1}{2}V_m$ or $(\gamma-\sigma)t$, and is equal to the angle through which the earth has turned relatively to the mean moon (at $14^\circ{\cdot}4920521$ per mean solar hour) since epoch.

* Fabricated times and heights would very likely be no worse than real observations during a few days of rough weather. A perfect tide table only claims to predict the tide apart from the influence of wind and atmospheric pressure; and, conversely, tidal observations must be sufficiently numerous to eliminate these influences by averages.

It would be advantageous to extend the table up to 90 days, but it can be used as it is for periods greater than 30 days by the division of the time into sets of 30 days. In the second period of 30 days 5°·7 must be *subtracted* from the tabular entry, for the third period 11°·4, and so on.*

For example: Find $\frac{1}{2}V_m$ for $78\frac{1}{2}^d$ 11^h 23^m. The day is $18\frac{1}{2}$ of the third 30, and the tabular entry for $18\frac{1}{2}^d$ 11^h is 113°·9, and subtracting 11°·4 we have 102°·5; 23^m gives 5°·6, so that $\frac{1}{2}V_m = 108°·1$. The correct result is 107°·99, and it is obvious that an error of 0°·1 may easily be incurred by the use of the table.

The row for day $-\frac{1}{2}$ is given because it may be necessary to use one tide before epoch; this row is used in the example below.

* Observe that the decimals run thus, ·0, ·5, ·0, &c., then ·4, ·9, ·4, &c., then ·8, ·3, ·8, &c., and so on. The first entry in which the sequence alters I call a "change." The incidence of "changes" may be found thus: $\gamma - \sigma$ is $14\frac{1}{2} - ·00795$; take Crelle's multiplication table for 795, and note where the last digit but three, having been 4, becomes 5; I say that this is a "change." For example, $559 \times 795 = 444405$, and $560 \times 795 = 445200$; then a change occurs at the 560th hour, or at the $(12 \times 46 + 8)^{th}$ hour, or at 23^d 8^h. If the table be continued to 60 and 90 days, &c., by subtracting 5°·7, 11°·4, &c., the changes will fall a little wrong, but they may easily be corrected by means of Crelle's table, as here shown.—(Added Aug. 2, 1890.)

Table of $(\gamma-\sigma)t$ or $\frac{1}{2}V_m$.

Days.	0h.	1h.	2h.	3h.	4h.	5h.	Days.	6h.	7h.	8h.	9h.	10h.	11h.
−½	186°·1	200°·6	215°·1	229°·6	244°·1	258°·6	−½	273°·0	287°·5	302°·0	316°·5	331°·0	345°·5
0	0	14·5	29·0	43·5	58·0	72·5	0	87·0	101·4	115·9	130·4	144·9	159·4
½	173·9	188·4	202·9	217·4	231·9	246·4	½	260·9	275·3	289·8	304·3	318·8	333·3
1	347·8	2·3	16·8	31·3	45·8	60·3	1	74·8	89·3	103·7	118·2	132·7	147·2
½	161·7	176·2	190·7	205·2	219·7	234·2	½	248·7	263·2	277·7	292·1	306·6	321·1
2	335·6	350·1	4·6	19·1	33·6	48·1	2	62·6	77·1	91·6	106·0	120·5	135·0
½	149·5	164·0	178·5	193·0	207·5	222·0	½	236·5	251·0	265·5	280·0	294·4	308·9
3	323·4	337·9	352·4	6·9	21·4	35·9	3	50·4	64·9	79·4	93·9	108·3	122·8
½	137·3	151·8	166·3	180·8	195·3	209·8	½	224·3	238·8	253·3	267·8	282·3	296·7
4	311·2	325·7	340·2	354·7	9·2	23·7	4	38·2	52·7	67·2	81·7	96·2	110·6
½	125·1	139·6	154·1	168·6	183·1	197·6	½	212·1	226·6	241·1	255·6	270·1	284·6
5	299·0	313·5	328·0	342·5	357·0	11·5	5	26·0	40·5	55·0	69·5	84·0	98·5
½	113·0	127·4	141·9	156·4	170·9	185·4	½	199·9	214·4	228·9	243·4	257·9	272·4
6	286·9	301·3	315·8	330·3	344·8	359·3	6	13·8	28·3	42·8	57·3	71·8	86·3
½	100·8	115·3	129·7	144·2	158·7	173·2	½	187·7	202·2	216·7	231·2	245·7	260·2
7	274·7	289·2	303·6	318·1	332·6	347·1	7	1·6	16·1	30·6	45·1	59·6	74·1
½	88·6	103·1	117·6	132·0	146·5	161·0	½	175·5	190·0	204·5	219·0	233·5	248·0
8	262·5	277·0	291·5	305·9	320·4	334·9	8	349·4	3·9	18·4	32·9	47·4	61·9
½	76·4	90·9	105·4	119·9	134·3	148·8	½	163·3	177·8	192·3	206·8	221·3	235·8
9	250·3	264·8	279·3	293·8	308·3	322·7	9	337·2	351·7	6·2	20·7	35·2	49·7
½	64·2	78·7	93·2	107·7	122·2	136·6	½	151·1	165·6	180·1	194·6	209·1	223·6
10	238·1	252·6	267·1	281·6	296·1	310·6	10	325·0	339·5	354·0	8·5	23·0	37·5
½	52·0	66·5	81·0	95·5	110·0	124·5	½	138·9	153·4	167·9	182·4	196·9	211·4
11	225·9	240·4	254·9	269·4	283·9	298·4	11	312·9	327·3	341·8	356·3	10·8	25·3
½	39·8	54·3	68·8	83·3	97·8	112·3	½	126·8	141·3	155·7	170·2	184·7	199·2
12	213·7	228·2	242·7	257·2	271·7	286·2	12	300·7	315·2	329·6	344·1	358·6	13·1
½	27·6	42·1	56·6	71·1	85·6	100·1	½	114·6	129·1	143·6	158·0	172·5	187·0
13	201·5	216·0	230·5	245·0	259·5	274·0	13	288·5	303·0	317·5	331·9	346·4	0·9
½	15·4	29·9	44·4	58·9	73·4	87·9	½	102·4	116·9	131·4	145·9	160·3	174·8
14	189·3	203·8	218·3	232·8	247·3	261·8	14	276·3	290·8	305·3	319·8	334·2	348·7
½	3·2	17·7	32·2	46·7	61·2	75·7	½	90·2	104·7	119·2	133·7	148·2	162·6

mins.	
1	0°·2
2	0·5
3	0·7
4	1·0
5	1·2
6	1·4
7	1·7
8	1·9
9	2·2
10	2·4
11	2·7
12	2·9
13	3·1
14	3·4
15	3·6
16	3·9
17	4·1
18	4·3
19	4·6
20	4·8
21	5·1
22	5·3
23	5·6
24	5·8
25	6·0
26	6·3
27	6·5
28	6·8
29	7·0

15	177·1	191·6	206·1	220·6	235·1	249·6	15	264·1	278·6	293·1	307·6	322·1	336·6	30	7·2
½	351·0	5·5	20·0	34·5	49·0	63·5	½	78·0	92·5	107·0	121·5	136·0	150·5	31	7·5
16	164·9	179·4	193·9	208·4	222·9	237·4	16	251·9	266·4	280·9	295·4	309·9	324·4	32	7·7
½	338·9	353·3	7·8	22·3	36·8	51·3	½	65·8	80·3	94·8	109·3	123·8	138·3	33	8·0
17	152·8	167·2	181·7	196·2	210·7	225·2	17	239·7	254·2	268·7	283·2	297·7	312·2	34	8·2
½	326·7	341·2	355·6	10·1	24·6	39·1	½	53·6	68·1	82·6	97·1	111·6	126·1	35	8·5
18	140·6	155·1	169·5	184·0	198·5	213·0	18	227·5	242·0	256·5	271·0	285·5	300·0	36	8·7
½	314·5	329·0	343·5	357·9	12·4	26·9	½	41·4	55·9	70·4	84·9	99·4	113·9	37	8·9
19	128·4	142·9	157·4	171·9	186·3	200·8	19	215·3	229·8	244·3	258·8	273·3	287·8	38	9·2
½	302·3	316·8	331·3	345·8	0·2	14·7	½	29·2	43·7	58·2	72·7	87·2	101·7	39	9·4
20	116·2	130·7	145·2	159·7	174·2	188·6	20	203·1	217·6	232·1	246·6	261·1	275·6	40	9·7
½	290·1	304·6	319·1	333·6	348·1	2·5	½	17·0	31·5	46·0	60·5	75·0	89·5	41	9·9
21	104·0	118·5	133·0	147·5	162·0	176·5	21	190·9	205·4	219·9	234·4	248·9	263·4	42	10·1
½	277·9	292·4	306·9	321·4	335·9	350·4	½	4·9	19·3	33·8	48·3	62·8	77·3	43	10·4
22	91·8	106·3	120·8	135·3	149·8	164·3	22	178·8	193·2	207·7	222·2	236·7	251·2	44	10·6
½	265·7	280·2	294·7	309·2	323·7	338·2	½	352·7	7·2	21·6	36·1	50·6	65·1	45	10·9
23	79·6	94·1	108·6	123·1	137·6	152·1	23	166·6	181·1	195·5	210·0	224·5	239·0	46	11·1
½	253·5	268·0	282·5	297·0	311·5	326·0	½	340·5	355·0	9·5	23·9	38·4	52·9	47	11·4
24	67·4	81·9	96·4	110·9	125·4	139·9	24	154·4	168·9	183·4	197·8	212·3	226·8	48	11·6
½	241·3	255·8	270·3	284·8	299·3	313·8	½	328·3	342·8	357·3	11·8	26·2	40·7	49	11·8
25	55·2	69·7	84·2	98·7	113·2	127·7	25	142·2	156·7	171·2	185·7	200·2	214·6	50	12·1
½	229·1	243·6	258·1	272·6	287·1	301·6	½	316·1	330·6	345·1	359·6	14·1	28·5	51	12·3
26	43·0	57·5	72·0	86·5	101·0	115·5	26	130·0	144·5	159·0	173·5	188·0	202·5	52	12·6
½	216·9	231·4	245·9	260·4	274·9	289·4	½	303·9	318·4	332·9	347·4	1·9	16·4	53	12·8
27	30·8	45·3	59·8	74·3	88·8	103·3	27	117·8	132·3	146·8	161·3	175·8	190·3	54	13·0
½	204·8	219·2	233·7	248·2	262·7	277·2	½	291·7	306·2	320·7	335·2	349·7	4·2	55	13·3
28	18·7	33·1	47·6	62·1	76·6	91·1	28	105·6	120·1	134·6	149·1	163·6	178·1	56	13·5
½	192·6	207·1	221·5	236·0	250·5	265·0	½	279·5	294·0	308·5	323·0	337·5	352·0	57	13·8
29	6·5	21·0	35·5	49·9	64·4	78·9	29	93·4	107·9	122·4	136·9	151·4	165·9	58	14·0
½	180·4	194·9	209·4	223·8	238·3	252·8	½	267·3	281·8	296·3	310·8	325·3	339·8	59	14·3
30	− 5·7														
60	−11·4														
90	−17·2														
120	−22·9														
150	−28·6														

To find V_s.

No table is necessary for the conversion of time into angle at 30° per hour to find V_s, or $2(\gamma-\eta)t$, since we multiply the hours by 30, and add half the number of minutes. This rule is the same for every day.

B. *The tides* S_2 *and* K_2.

It is required to compute U and ϕ from

$$U \cos\phi = \Pi + \lambda_n f'' \cos\omega,$$

$$U \sin\phi = \lambda_n f'' \sin\omega,$$

where

$$\omega = 2h_0 - 2\nu'' + \alpha_n,$$

and

$$\Pi = 1 + 3\left(\frac{\text{sun's parx.} - \text{mean parx.}}{\text{mean parx.}}\right),$$

the sun's parallax referred to being its value at the middle of the period under reduction.

If, for example, February 14 is the middle of the period, Π is found thus:—

Sun's parx. Feb. 14 = $8''\cdot95$, mean parx. = $8''\cdot85$, diff. = $+0''\cdot10$.

Then

$$\Pi = 1 + \frac{3 \times 0\cdot10}{8\cdot85} = 1\cdot034.$$

The period under reduction consists in this case of an exact number of semi-lunations. The following table gives λ_n and α_n, according to the number of semi-lunations:—

No. of semi-lunations.	1.	2.	3.	4.	5.	6.
$\log \lambda_n$	9·4300	9·4159	9·3920	9·3575	9·3113	9·2517
α_n	14°·28	28°·82	43°·36	57°·90	72°·43	86°·97

h_0 is the sun's mean longitude at epoch, found from Naut. Al.; and $2\nu''$, f'' are found from Baird's Manual in the tables applicable to the tide K_2.

C. *The Tides* N *and* L.

Summations are carried out over quarter-lunar-anomalistic periods, numbered i, ii, iii, &c. Grand totals are then made in two different ways, viz.:—

$$[\Sigma(\text{i}+\text{ii}) - \Sigma(\text{iii}+\text{iv})] + [\Sigma(\text{v}+\text{vi}) - \Sigma(\text{iii}+\text{iv})]$$
$$+ [\Sigma(\text{v}+\text{vi}) - \Sigma(\text{vii}+\text{viii})] + \&\text{c., to find } S^\circ,$$

and

$$[\Sigma(\text{ii}+\text{iii}) - \Sigma(\text{iv}+\text{v})] + [\Sigma(\text{vi}+\text{vii}) - \Sigma(\text{iv}+\text{v})]$$
$$+ [\Sigma(\text{vi}+\text{vii}) - \Sigma(\text{viii}+\text{ix})] + \&\text{c., to find } S^{\frac{1}{2}\pi},$$

where, for example, Σ(i+ii) denotes summation carried over the half period made up of i and ii. These totals are multiplied by certain mean cosines and sines (whose values are given in F), and are summed. The next process is multiplication by a factor Φ ($\pi/4(n+1)(m-1)$ of § 4), of which the value depends on the number of quarter-lunar-anomalistic periods under treatment. The following table gives the value of this factor :—

No. of ¼-lunar-anom. periods.	iii.	v.	vii.	ix.	xi.	xiii.
Φ	0·02950	0·01475	0·00738	0·00492	0·00369	0·00295

The angle j is also required; it depends on the time of the first tide under reduction. If t_o be the time in hours since epoch to the first tide,

$$j = 1^\circ{\cdot}690 - 0^\circ{\cdot}5444\ t_o.$$

For instance, in the example below the first tide is at $3^h\ 14^m$ of day $-\frac{1}{2}$; this is $8^h\ 46^m$, or $8^h{\cdot}77$, *before* epoch, so that $t_o = -8^h{\cdot}77$; then

$$j = 1^\circ{\cdot}690 + 0^\circ{\cdot}5444 \times 8{\cdot}77 = +6^\circ{\cdot}46.$$

D. *The Tides* K_1, O, P.

Summations are carried out over quarter-lunar periods numbered I, II, III, &c., and totals are formed like those mentioned in C, and a factor Ψ (which differs slightly from Φ) is required in the formation of S° and $S^{\frac{1}{2}\pi}$. This factor depends on the number of quarter-lunar periods under treatment, and the following table gives its value :—

No. of ¼-lunar periods.	III.	V.	VII.	IX.	XI.	XIII.
Ψ	0·02976	0·01488	0·00744	0·00496	0·00372	0·00298

The angles i and l are required; they depend on the time of the first tide under reduction. If t_o be the time in hours of the first tide since epoch,

$$i = 1^\circ{\cdot}705 - 0^\circ{\cdot}549\ t_o.$$

$$l = -0^\circ{\cdot}255 + 0^\circ{\cdot}082\ t_o.$$

For instance, in the example below we have, as shown in C, $t_o = -8^h{\cdot}77$, and

$$i = +6^\circ{\cdot}52,$$

$$l = -0^\circ{\cdot}97.$$

It is required to compute T and ψ from

$$T \cos \psi = f' - \rho_n \cos \theta,$$

$$T \sin \psi = \rho_n \sin \theta,$$

where $$\theta = 2h_0 - \nu' + l + \beta_n.$$

The period under reduction consists in this case of an exact number of quarter-lunar periods, and the following table gives the values of ρ_n and β_n, according to the number of quarter-lunar periods :—

No. of $\frac{1}{4}$-lunar periods.	III.	V.	VII.	IX.	XI.	XIII.
$\log \rho_n$	9·5749	9·5628	9·5508	9·5303	9·5009	9·4618
β_n	20°·20	33°·66	47°·13	60°·59	74°·06	87°·52

h_0 is the sun's mean longitude at epoch, the formula for l is given above, and ν', f′ are found from Baird's Manual in the tables applicable to the tide K_1.

E. *Rules for the Partition of the Observations into Groups.*

If the first event after epoch is a L.W., either omit it from the reductions, or let the first tide be the H.W. which precedes epoch. Thus we are to begin with a H.W.*

The H.W. and L.W. are treated apart in separate tables.

Each tide (H.W. or L.W., as the case may be) is numbered consecutively, from 1 onwards.

The following are rules for partitions :—

* This is not necessary, but it makes the statements of the subsequent rules simpler, as they have not to be given in an alternative form.

For Tides N and L.

Quarter-lunar-anomalistic periods, numbered i, ii, iii, &c.

No. of ¼-lunar-anomalistic periods	i.	ii.	iii.	iv.	v.	vi.	vii.	viii.	ix.	x.	xi.	xii.	xiii.
No. of last H.W. in the ¼ period .	14	27	40	53	67	80	93	107	120	133	147	160	173
No. of last L.W. in the ¼ period ..	13	26	40	53	66	80	93	106	120	133	146	159	173
Total No. of tides up to end of each ¼ period	27	53	80	106	133	160	186	213	240	266	293	319	346

For Tides K_1, O, P.

Quarter-lunar periods, numbered I, II, III, &c.

No. of ¼-lunar periods	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.
No. of last H.W. in the ¼ period ..	13	27	40	53	66	79	93	106	119	132	145	159	172
No. of last L.W. in the ¼ period ..	13	26	39	53	66	79	92	105	119	132	145	158	171
Total No. of tides up to end of each ¼ period	26	53	79	106	132	158	185	211	238	264	290	317	343

For M_2 and S_2.

Semi-lunations numbered 1, 2, 3, &c.

No. of semi-lunation	1	2	3	4	5	6
No. of last H.W. in the semi-lunation	29	57	86	114	143	171
No. of last L.W. in the semi-lunation	28	57	85	114	142	171
Total No. of tides up to end of each semi-lunation	57	114	171	228	285	342

For Annual and Semi-annual Tides.

Months, or $\frac{1}{12}$th parts of a year, numbered 1, 2, 3.

No. of month	1	2	3
No. of last H.W. in month	59	118	176
No. of last L.W. in month	59	117	176
Total No. of tides up to end of each month.	118	235	352
No. of tides in each month	118	117	117

The epoch for the second quarter year should be 91 days after first epoch, that for the third 92 days after the first, for the fourth 91 days after the third, except in leap year, when the last should also be 92 days.

There are six tides (or about thirty-seven hours) more in a quarter year than in xiii quarter-lunar-anomalistic periods; the times of these six tides (or ten tides in one of the quarters) are to be omitted from the reduction, and their heights are only required when the annual or semi-annual tides are to be found.

F. *Cosine and Sine Factors for all the Tides.*

These are the cosines and sines of 2° 30′, 7° 30′, 12° 30′, &c. They are as follows:—

Cosine and Sine Factors.

Read downwards for cosines, upwards for sines.

1. 0·999	10. 0·676
2. 0·991*	11. 0·609*
3. 0·976	12. 0·537
4. 0·954	13. 0·462
5. 0·924*	14. 0·383*
6. 0·887	15. 0·301
7. 0·843	16. 0·216
8. 0·793*	17. 0·130*
9. 0·737	18. 0·044

In the evaluation of $S^{\circ} \frac{\cos}{\sin} \frac{3}{2} V_m$ and $S^{\frac{1}{4}\pi} \frac{\cos}{\sin} \frac{3}{2} V_m$, only the factors marked * are required.

G. *Increments of Arguments in Various Times.*

The following table gives the increments of arguments of the several tides in various periods, multiples of 360° being subtracted. This table facilitates verification of the calculation of the harmonic constants.

	M_2.	S_2.	K_2.	N.
1 hour.....	28°·984104	30°·00000	30°·08214	28°·43973
1 day......	− 24 ·3815	0	+ 1 ·9713	− 37 ·4465
10 days.....	+116 ·185	0	+ 19 ·713	− 14 ·465
100 days	+ 81 ·85	0	−162 ·87	−144 ·65
	L.	K_1.	O.	P.
1 hour.....	29°·52848	15°·04107	13°·94304	14°·95893
1 day......	− 11 ·3165	+ 0 ·9856	− 25 ·3671	− 0 ·9856
10 days.....	−113 ·165	+ 9 ·856	+106 ·329	− 9 ·856
100 days	− 51 ·65	+98 ·56	− 16 ·71	− 98 ·56

Example.

(a.) *Place, Time, Datum Level, and Unit of Length.*

The case chosen is three months of observation (in reality the tidal predictions of the Indian Government) at Bombay, and the epoch is 0^h, January 1, 1887.

A datum at or very near mean water-mark is taken, so that all the H.W. are positive and the L.W. negative. This datum is found by taking the mean of all the H.W. and L.W. of the original observations. In this case 99 inches was subtracted from all the tide heights. I might more advantageously have subtracted 102 or 103 inches, but 99 inches was chosen from considerations applicable to my earlier attempts, but which do not apply to the computation in its present form.

At places where there is a large annual inequality in the height of water, it would be advisable to use a different datum for each quarter of a year. It is not, however, important that the datum should conform rigorously to mean water-mark, for even the discrepancy of 3½ inches, which occurs in my example, does not materially affect the result.

In recording the heights, a convenient unit of length is to be used, and it is advantageous that the H.W. and the L.W. should be expressible by two figures, so that the larger H.W. and L.W. shall fall into the eighties and nineties. The unit of length is here the inch.

(b.) *Times and Angles.*

The times of H.W., numbered consecutively, are entered in a table, as shown on p. 313. Since 0^h astronomical time is the epoch, the P.M. tides will come in the half days which are numbered with integrals, and the A.M. tides in the half days which fall between the integral numbers.

From time to time there will be a half day with no H.W.; this row in the table should be left blank, but there happens to be no such row in the sample shown. A computation form for times and angles might be printed, for, although the exigencies of the printer have not allowed the entries to be equally spaced in the sample below, yet the computation form might be printed with equal spaces, and the dividing lines are to be filled in by hand.

The L.W. table is similar.

Both H. and L.W. are to be divided into quarter-lunar-anomalistic and quarter-lunar periods, and semi-lunations, according to the rules given in E. These partitions and the numbering of the entries could not be printed, because of the occasional blank rows.

The formation of $\frac{1}{2}V_m$ and of V_s, by means of Table A and the rule following it, is obvious. In the subtractions and additions under the headings $\frac{1}{2}V_m - V_s$ and $\frac{1}{2}V_m + V_s$, 360° is added or subtracted where necessary. V_m is found by doubling $\frac{1}{2}V_m$.

(c.) *The Heights.*

The H.W. heights are written in columns (with the same blanks as in the table of times and angles), and are so arranged, either on strips of paper, or by folding the paper, that the heights may be pinned to the times, bringing each height opposite to an angle on the same row with the time corresponding to that height. The heights will on one occasion have to be pinned opposite the V_m column, on a second occasion opposite the V_s column, and on a third occasion opposite the $\frac{1}{2}V_m$ column.

The L.W. heights are written in similar columns, but the minus signs should be omitted.

It is well to divide the columns, or to put fiducial marks in the table for easy verification of the proper allocation of the heights with the times. Any marks suffice, but the division into quarter-anomalistic periods, as shown below, seems to be as good as any other.

If it is proposed to evaluate the annual and semi-annual tides, it is necessary to carry on the heights beyond the times by 3 (or 5) H.W. and 3 (or 5) L.W., and to partition them into months. The mean for each month is evaluated, and if the successive quarters of the year are referred to different data, the mean monthly heights must all be referred to a common datum. This process is not carried out

H.W.

No. of entry.	Day.	Time.	min.	hr.	$\frac{1}{2}V_m$.	min.	hr.	V_s.	$\frac{1}{2}V_m - V_s$.	$\frac{1}{2}V_m + V_s$.	V_m.	½-Anom. periods.	½-Lunar periods.	Semi-lunai ons.
		h. m.												
1	−½	3 14	3·4	229·6	233·0	7·0	90	97·0	136·0	330·0	106·0	i begins	I begins	1 begins
2	0	3 38	9·2	43·5	52·7	19·0	90	109·0	303·7	161·7	105·4			
3	½	3 49	11·8	217·4	229·2	24·5	90	114·5	114·7	343·7	98·4			
4	1	4 31	7·5	45·8	53·3	15·5	120	135·5	277·8	188·8	106·6			
5	½	4 30	7·2	219·7	226·9	15·0	120	135·0	91·9	1·9	93·8			
6	2	5 46	11·1	48·1	59·2	23·0	150	173·0	246·2	232·2	118·4			
7	½	5 20	4·8	222·0	226·8	10·0	150	160·0	66·8	26·8	93·6			
8	3	7 15	3·6	64·9	68·5	7·5	210	217·5	211·0	286·0	137·0			
9	½	6 29	7·0	224·3	231·3	14·5	180	194·5	36·8	65·8	102·6			
10	4	8 28	6·8	67·2	74·0	14·0	240	254·0	180·0	328·0	148·0			
11	½	7 40	9·7	226·6	236·3	20·0	210	230·0	6·3	106·3	112·6			
12	5	9 22	5·3	69·5	74·8	11·0	270	281·0	153·8	355·8	149·6			
13	½	8 47	11·4	228·9	240·3	23·5	240	263·5	36·8	143·8	120·6		I ends	
14	6	10 9	2·2	71·8	74·0	4·5	300	304·5	129 5	18·5	148·0	i ends	II begins	
15	½	9 43	10·4	231·2	241·6	21·5	270	291·5	310·1	173·1	123·2	ii begins		
	&c.			&c.			&c.		&c.	&c.	&c.		&c.	
169	½	1 22	5·3	220·0	225·3	11·0	30	41·0	184·3	266·3	90·6			
170	87	2 7	1·7	48·4	50·1	3·5	60	63·5	346·6	113·6	100·2			
171	½	1 53	12·8	207·9	220·7	26·5	30	56·5	164·2	277·2	81·4			6 ends
172	88	2 49	11·8	36·2	48·0	24·5	60	84·5	323·5	132·5	96·0		XIII ends	
173	½	2 30	7·2	210·1	217·3						74·6	xiii ends		

in my example, because it is useless to attempt the evaluation of these tides of long period from three months of observation.

H.W.
55 i
23
47
19
41
19
37
26
35
37
37
52
42
66 i
50 ii
80
57
91
63
None
99
67
101
66
99
65
92
58 ii
81 iii
&c., &c.

The sum of all the H.W. entries to the end of xiii is 9791. The sum of all the L.W. entries to the end of xiii is 8577. There are 173 H.W. and 173 L.W. Hence mean sea-level is $\frac{1}{346}$ (9791—8577), which is equal to +3·51. It would have been better, therefore, to have subtracted 103 inches instead of 99 inches from all the heights. This would have given mean sea level at −0·49 from the datum adopted.

(d.) *Sorting the Heights according to the Values of the Angles.*

In the tables below the column 0° belongs to all angles between 0° and 5°; 5° belongs to 5° to 10°; and so on.

Where an angle falls *exactly* on a multiple of 5°, an arbitrary rule of classification is required, and it is easiest to deem it to belong to the next succeeding 5°, rather than to the preceding 5°.

(e.) *Sorting according to Values of* V_m.

The H. and L.W. are treated in separate tables, similar in form save that the − signs of the L.W. heights are omitted.

The sheets of heights (c) are pinned opposite to the V_m's on the tables of angles (b), and the heights are then entered successively into the columns corresponding to their V_m's in a table like the following.

The division into quarter-lunar-anomalistic periods is maintained, but as this sorting is to serve a double purpose, it is necessary to mark the end of the last semi-lunation. In these tables there are two H.W. and two L.W., which fall after the end of 6 semi-lunations, and before the end of xiii quarter-lunar-anomalistic periods.

Nearly all the entries fall into one quadrant for H.W., and into another for L.W. Thus there are no H.W. entries in the 4th quadrant, and no L.W. entries in 2nd quadrant; there are altogether only 10 H.W. entries in 1st quadrant, and 3 in 3rd quadrant; and there are only 4 L.W. entries in the 1st quadrant, and 17 in 3rd quadrant. Something like this would hold true at all ports.

H.W. V_m.

		Angle	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	No. of entries.
1st quad.	v		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	39	45	53	4
			.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	34	
	ix		.	.	.	.	.	.	.	.	.	.	.	.	.	.	38	45	53	.	3
	x		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	31	.	.	1
	xiii		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	50	.	1
			6 semi-lunations end																		
			.	.	.	.	.	.	.	.	.	.	.	.	.	.	42	.	.	.	1
2nd quad.		Angle	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175	
	i		41	47	35	55	37	19	42	.	.	26	.	37	.	.	.	.	.	.	14
			37	.	.	23	.	.	.	.	.	.	.	52	.	.	.	.	.	.	
			.	.	.	19	.	.	.	.	.	.	.	66	.	.	.	.	.	.	
	ii		.	.	.	.	.	66	50	99	99	91	80	.	.	.	.	.	.	.	13
			.	.	.	.	.	65	57	.	101	.	.	.	.	.	.	.	.	.	
			.	.	.	.	.	58	63	.	.	.	.	.	.	.	.	.	.	.	
			.	.	.	.	.	.	67	.	.	.	.	.	.	.	.	.	.	.	
			.	.	.	.	.	.	92	.	.	.	.	.	.	.	.	.	.	.	
	iii &c.			&c.					&c.				&c.					&c.			&c.

	xi	.	.	.	45	61	76	96	85	99	.	.	.	.	39	.	.	.	.	14
		.	.	.	.	30	.	88	79	87	.	.	.	.	.	.	.	.	.	
		.	.	.	.	.	.	.	69	45	.	.	.	.	.	.	.	.	.	
		.	.	.	.	.	.	.	57	.	.	.	.	.	.	.	.	.	.	
	xii	.	.	.	.	.	.	.	.	20	.	.	56	66	51	20	44	27	42	11
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	64	.	36	.	
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	61	.	
	xiii	57	.	61	59	.	65	.	67	.	60	67	.	.	.	.	.	.	.	11
		.	.	55	.	.	62	.	62	.	.	.	.	.	.	.	.	.	.	
		6 semi-lunations end																		
		.	49	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	
3rd quad.	Angle	180	185	190	195	200	205	210	215	220	225	230	235	240	245	250	255	260	265	
	viii	56	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
	xii	49	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
		55	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	
		6 semi-lunations end																		
																			Total	173
4th quad.		Nil																		

L.W. V_m.

		Angle 0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	No. of entries.
1st quad.	viii	30	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
	xii	36	19	28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3
		......6 semi-lunations end......																		
2nd quad.		Nil.																		
3rd quad.		Angle 180	185	190	195	200	205	210	215	220	225	230	235	240	245	250	255	260	265	
	i	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	26	16	.	2
	v	.	.	.	.	.	.	.	.	.	.	.	.	.	22	32	.	41	.	4
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	12	.	.	.	
	ix	.	.	.	.	.	.	.	.	.	.	.	.	.	26	35	48	51	44	7
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	15	.	.	45	
	xiii	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	36	.	.	4
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	62	.	.	
		......6 semi-lunations end......																		
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	27	.	.	
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	56	.	.	
4th quad.		Angle 270	275	280	285	290	295	300	305	310	315	320	325	330	335	340	345	350	355	
	&c.		&c.				&c.					&c.					&c.			&c.
	xii	.	.	.	.	.	.	.	.	.	.	.	55	.	42	50	13	..	44	10
		.	.	.	.	.	.	.	.	.	.	.	55	.	53	.	44	.	.	
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	47	.	.	
		.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	50	.	.	
	xiii	67	50	.	66	54	.	55	.	60	56	.	.	.	.	.	.	.	.	8
		.	.	.	.	.	.	63	.	.	.	.	.	.	.	.	.	.	.	
		......6 semi-lunations end......																		Total 173

(f.) *Table of Sums for* N *and* L. (See p. 320.)

Maintaining the divisions i, ii, iii, &c., it is now necessary to sum each of the four times 18 vertical columns in each of the xiii divisions, to subtract the 18 columns of the 3rd quadrant from the 18 columns of the 1st, and to subtract the 18 columns of the 4th quadrant from the 18 of the 2nd. The 2nd−4th columns have then to be reversed.

Since in this case nearly all the H.W. fall in the 2nd quadrant, and nearly all the L.W. fall in the 3rd quadrant, it is easy to write down at once 2nd−4th, and 1st−3rd, as shown on the next page.

In this the − signs of the L.W. entries have to be reintroduced, but as the L.W. lie mostly in 3rd quadrant, which enters with negative sign, they become positive again. It thus happens that nearly all the columns come out + ; there are, however, a few − in xii.

(g.) *Table of Sums for* M_2. (See p. 321.)

We now disregard the sub-divisions i, ii, iii, &c., and sum the 4 times 18 columns into grand totals, stopping the summations, however, at the end of 6 semi-lunations (*i.e.*, at 171 H.W. and 171 L.W.).

It would hardly be wise to attempt in this case the subtractions 1st−3rd, 2nd−4th, without the intermediate steps.

The following table (p. 321) gives the results.

(h.) *General Rule for Cosine and Sine Summations.*

For 'cosines' the 18 numbers required are derived from (1st−3rd)—(2nd−4th, reversed).

For 'sines' the 18 numbers required are derived from (1st−3rd) + (2nd−4th, reversed).

(i.) *Evaluations of* N *and* L (continued). (See p. 322.)

Cosines.—From Table (f) of Sums enter the 18 'cosine' numbers, in accordance with (h) in xiii vertical columns, and perform the operations indicated in the example on page 322.

The column of 'cosine factors' are those given in F, and Φ is given in C for xiii ½-lunar-anomalistic periods.

(f.) *Sums for* N *and* L.

	(2nd – 4th) H.W.	78	47	35	97	37	19	42	.	.	26	.	155	.	.	.	.	.	.
	(2nd – 4th) L.W.	9	.	.	.	62	40	216	11	18	.	.	.	.	.	.	.	.	.
	(2nd – 4th) Total	87	47	35	97	99	59	258	11	18	26	.	155	.	.	.	.	.	.
	(1st – 3rd) H.W.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	(1st – 3rd) L.W.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	26	16	.
	(1st – 3rd) Total	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	26	16	.
	(2nd – 4th) Reversed........	.	.	.	.	.	.	155	.	26	18	11	258	59	99	97	35	47	87
	&c.	&c.						&c.					&c.				&c.		
xii	2nd – 4th H.W.	.	.	.	.	.	.	.	.	20	.	.	56	66	51	84	44	124	42
	2nd – 4th L.W.	.	.	.	.	.	.	.	.	.	.	.	110	.	95	50	154	.	44
	(2nd – 4th) Total	.	.	.	.	.	.	.	.	20	.	.	166	66	146	134	198	124	86
	1st – 3rd H.W.	−104	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	1st – 3rd L.W.	−36	−19	−28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	(1st – 3rd) Total..........	−140	−19	−28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	(2nd – 4th) Reversed	86	124	198	134	146	66	166	.	.	20	.	.	.	.	.	.	.	.
xiii	2nd – 4th H.W.	57	49	116	59	.	127	.	129	.	60	67	.	.	.	.	.	.	.
	2nd – 4th L.W.	67	50	.	66	54	.	118	.	60	56	.	.	.	.	.	.	.	.
	(2nd – 4th) Total	124	99	116	125	54	127	118	129	60	116	67	.	.	.	.	.	.	.
	1st – 3rd H.W.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	42	.	50	.
	1st – 3rd L.W.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	181	.	.
	(1st – 3rd) Total	..	..	..	..	..	..	..	..	..	..	..	..	..	..	42	181	50	..
	(2nd – 4th) Reversed	..	..	..	..	..	..	..	67	116	60	129	118	127	54	125	116	99	124

(g.) *Sums for* M_2.

1st H.W. ...	.	.	.	.	.	.	.	.	.	.	.	.	.	.	38	115	148	87
1st L.W. ...	−66	−19	−28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1st Total	−66	−19	−28	.	.	.	.	.	.	.	.	.	.	.	38	115	148	87
2nd H.W. ...	262	247	210	452	292	976	1056	1120	941	678	598	696	439	135	395	122	317	216
2nd L.W. ...	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2nd Total ...	262	247	210	452	292	976	1056	1120	941	678	598	696	439	135	395	122	317	216
3rd H.W. ...	160	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3rd L.W. ...	.	.	.	.	.	.	.	.	.	.	.	.	.	−48	−94	−172	−108	−199
3rd Total ...	160	.	.	.	.	.	.	.	.	.	.	.	.	−48	−94	−172	−108	−199
4th H.W. ...	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4th L.W. ...	−264	−188	−54	−481	−548	−810	−1453	−1095	−532	−518	−436	−240	−353	−220	−209	−241	−52	−66
4th Total ...	−264	−188	−54	−481	−548	−810	−1453	−1095	−532	−518	−436	−240	−353	−220	−209	−241	−52	−66
(2nd−4th) Total....	526	435	264	933	840	1786	2509	2215	1473	1196	1034	936	792	355	604	363	369	282
(1st−3rd) Total	−226	−19	−28	.	.	.	.	.	.	.	.	.	.	48	132	287	256	286
(2nd−4th) Reversed.	282	369	363	604	355	792	936	1034	1196	1473	2215	2509	1786	840	933	264	435	526

Cosines.					Sums in pairs.				Differences.					Sum of 5 preceding cols.	× Cosine factors.		
i	ii	&c.	xii	xiii	i + ii	iii + iv	&c.	xi + xii	i + ii − (iii + iv)	v + vi − (iii + iv)	v + vi − (vii + viii)	ix + x − (vii + viii)	ix + x − (xi + xii)			+	−
.	.		−226	.	.	− 62		−226	+ 62	+ 62	+220	+220	+226	+ 790 ×	·999	= 790	.
.	.		−143	.	.	−123		−143	+123	+123	+122	+122	+143	+ 633	·991	628	.
.	.		−226	.	.	−165	&c.	−226	+165	+165	.	.	+226	+ 556	·976	542	.
.	.	&c.	−134	.	.	−182		−134	+182	+182	+288	+288	+134	+1074	·954	1025	.
.	.		−146	.	.	.		−185	.	.	+170	+170	+185	+ 525	·924	485	.
.	.		− 66	.	.	−410		− 66	+410	+410	+316	+316	+ 66	+1518	·887	1346	.
−155	.		−166	.	−155	−182	&c.	−166	+ 27	− 11	−139	−132	− 20	− 275	·843	.	232
.	− 80	&c.	.	− 67	− 80	−334		− 15	+254	+207	+114	+ 71	−155	+ 491	·793	389	.
− 26	− 91		.	−116	−117	−178		− 46	+ 61	− 39	−106	−300	−365	− 749	·737	.	551
− 18	−311		− 20	− 60	−329	−235	&c.	−336	− 94	+ 55	+ 11	+ 49	+194	+ 215	·676	145	.
− 11	−211		.	−129	−222	−166		−378	− 56	−567	−654	−429	−130	−1836	·609	.	1118
−258	−529	&c.	.	−118	−787	−225		−479	−562	− 27	+310	+476	+393	+ 590	·537	317	.
− 59	−371		.	−127	−430	−394		−350	− 36	+179	− 68	+ 24	+227	+ 326	·462	151	.
− 99	− 93		.	− 54	−192	.		− 91	−192	− 69	+233	+218	+ 7	+ 197	·383	75	.
− 97	−163		.	− 83	−260	−163	&c.	− 45	− 97	+ 58	−105	−103	− 58	− 305	·301	.	92
− 9	.	&c.	.	+ 65	− 9	.		.	− 9	− 20	− 20	+ 70	+ 70	+ 91	·216	20	.
− 31	.		.	− 49	− 31	.		.	− 31	− 94	− 94	− 54	− 54	− 327	·130	.	43
− 87	.		.	−124	− 87	.		.	− 87	− 70	− 70	− 69	− 69	− 365	·044	.	16
																+5913	−2052
																−2052	
														Total.....		+3861	
														factor Φ...	×	·00295	
														$S^{\circ}/_{\circ} \cos V_m$	=	+11·38	

Cosines (continued).—Form columns ii+iii, iv+v, vi+vii, viii+ix, x+xi, xii+xiii. Form difference columns (ii+iii)−(iv+v), (vi+vii) − (iv+v), (vi+vii) − (viii+ix), (x+xi) − (viii+ix), (x+xi)−(xii+xii); add the 5 difference columns together; multiply by cosine factors; sum and multiply by Φ or 0·00295.

The result is :—

$$S^{\frac{1}{2}\pi} h \cos V_m = -8{\cdot}38.$$

Sines.—From Table (f) of Sums, enter 18 "sine" numbers, in accordance with (h) in xiii vertical columns.

Perform all the same operations as those on "cosine" numbers, save that we use sine factors, which are the same as cosine factors in inverse order, viz., beginning with 0·044 and ending with 0·999.

The two results are—

$$S^{\circ} h \sin V_m = +5{\cdot}94, \qquad S^{\frac{1}{2}\pi} h \sin V_m = +10{\cdot}06.$$

Collecting results, proceed thus :—

$S^{\circ} h \cos V_m$	$= +11{\cdot}38.$	$S^{\frac{1}{2}\pi} h \cos V_m$	$= -\ 8{\cdot}38.$
$S^{\frac{1}{2}\pi} h \sin V_m$	$= +10{\cdot}06.$	$S^{\circ} h \sin V_m$	$= +\ 5{\cdot}94.$
Sum	$= +21{\cdot}44.$	Sum	$= -\ 2{\cdot}44.$
Diff.	$= +\ 1{\cdot}32.$	Diff.	$= -14{\cdot}32.$
$P = \frac{1}{2}$ sum	$= +10{\cdot}72.$	$R = \frac{1}{2}$ sum	$= -\ 1{\cdot}22.$
$Q = \frac{1}{2}$ diff.	$= +\ 0{\cdot}66.$	$S = \frac{1}{2}$ diff.	$= -\ 7{\cdot}16.$

(i.) *Evaluation* M_2 (continued).

From the Table (g) of Sums for M_2 enter in one vertical column 18 cosine numbers, in accordance with (h); multiply them by cosine factors; add up and divide by the total number of entries for 6 semi-lunations, viz., 342.

The result is :—

$$A_m = \tfrac{1}{342} \Sigma h \cos V_m = -30{\cdot}58.$$

Then enter in vertical column 18 sine numbers, in accordance with (h); multiply them by sine factors, add up, and divide by 342.

The result is :—

$$B_m = \tfrac{1}{342} \Sigma h \sin V_m = +38{\cdot}47.$$

(k.) *Sorting according to Values of* V_s, *and Evaluation of* S_2, K_2.

The H. and L.W. are treated in separate tables, similar in form save that the − signs of the L.W. heights are omitted.

The sheets of heights (c) are pinned opposite to the V_s's on the Tables of Angles (b), and the heights are entered successively into the columns corresponding to their V_s's in a table like (e), which was used for sorting according to values of V_m. The sorting is carried as far as the end of an exact multiple of a semi-lunation,—in this case to the end of 6 semi-lunations. No sub-division is *necessary*, but for the purpose of verification it is useful to break the entries into groups of about 40. This is conveniently done by a division after each third $\frac{1}{4}$-lunar-anomalistic period, so that i, ii, iii would be the first group; iv, v, vi the second; vii, viii, ix the third; and x, xi, xii, and all but the end of xiii, the last.

In this case the entries fall into all the four quadrants with about equal frequency.

We next sum the four times 18 columns, just as with M_2 in (g), and form $1^{st}-3^{rd}$ and $2^{nd}-4^{th}$, reversed, in the same way.

Next we write the 18 cosine numbers, $(1^{st}-3^{rd})-(2^{nd}-4^{th}$, reversed) in vertical column, multiply by cosine factors, add, and divide by the total number of entries, which is 342. Afterwards write the sine-numbers $(1^{st}-3^{rd})+(2^{nd}-4^{th}$, reversed), multiply by sine factors, add, and divide by 342.

The results are:—

$$A_s = \frac{1}{342}\Sigma h \cos V_s = +21{\cdot}08. \qquad B_s = \frac{1}{342}\Sigma h \sin V_s = +3{\cdot}62.$$

(l.) *Sorting according to Values of $\frac{1}{2}V_m$.*

The whole process is precisely parallel to the sorting according to values of V_m in (e); the thirteen divisions are, however, given by the quarter-lunar-periods I, II, ... XIII. The only difference lies in the substitution of the factor Ψ (for XIII equal to 0·00298) for Φ. It is unnecessary to give an example.

The results are:—

$$S^{\circ}h \cos \tfrac{1}{2}V_m = -10{\cdot}50, \qquad S^{\circ}h \sin \tfrac{1}{2}V_m = +8{\cdot}04,$$
$$S^{\frac{1}{2}\pi}h \cos \tfrac{1}{2}V_m = +\ 0{\cdot}40, \qquad S^{\frac{1}{2}\pi}h \sin \tfrac{1}{2}V_m = +3{\cdot}74.$$

(m.) *Sorting of $\frac{1}{2}V_m$.*

It is required to find what the sums in (l) would be if every H.W. height had been unity, and every L.W. the same both in magnitude *and sign;* in fact to find $S^{\circ} \cos \frac{1}{2}V_m$, $S^{\frac{1}{2}\pi} \cos \frac{1}{2}V_m$, &c.

This is done by counting the entries in the preceding sorting in (l) without regard to magnitude, taking the L.W. entries as actually positive, instead of being (as they are) negative quantities with the negative sign suppressed.

Since in this case we have simply to count entries which are all treated as positive, the table of sums of H. and L.W. may be written together. The following example gives part of the work—

H. and L.W. $\frac{1}{2}V_m$.

I.																		
2nd	.	.	.	.	.	.	.	1	1	1	.	1	1	1	.	.	.	.
4th	.	.	.	.	.	.	.	.	.	.	.	3	4	.	.	.	.	.
2nd – 4th......	.	.	.	.	.	.	.	+1	+1	+1	.	−2	−3	+1	.	.	.	.
1st	.	.	.	.	.	.	.	.	.	.	2	1	.	1	2	.	.	.
3rd	.	.	.	.	.	.	.	.	.	3	2	1	1	.	.	.	.	.
1st – 3rd	.	.	.	.	.	.	.	.	.	−3	0	0	−1	+1	+2	.	.	.
2nd – 4th, rev.	.	.	.	.	+1	−3	−2	.	+1	+1	+1	.	.	.	.	.	.	.
II.																		
2nd	.	.	.	.	.	.	.	.	.	.	.	.	4	3	.	.	.	.
4th	.	.	.	.	.	.	.	.	.	.	2	3	1	.	.	.	.	.
2nd – 4th......	.	.	.	.	.	.	.	.	.	.	−2	−3	+3	+3	.	.	.	.
1st	.	.	.	.	.	.	.	.	.	.	.	.	2	3	2	.	.	.
3rd	.	.	.	.	.	.	.	.	.	.	.	3	4	.	.	.	.	.
1st – 3rd	.	.	.	.	.	.	.	.	.	.	.	−3	−2	+3	+2	.	.	.
2nd – 4th, rev.	.	.	.	.	+3	+3	−3	−2	.	.	.	.	.	.	.	.	.	.
III.		&c.							&c.						&c.			

We next proceed to form XIII columns of cosine numbers, and generally to operate exactly as though these numbers were heights; and then proceed with XIII columns of sine numbers in the same way.

The results are

$$S^{\circ} \cos \tfrac{1}{2}V_m = -0{\cdot}0522, \qquad S^{\circ} \sin \tfrac{1}{2}V_m = +0{\cdot}0117,$$

$$S^{\frac{1}{2}\pi} \cos \tfrac{1}{2}V_m = -0{\cdot}0168, \qquad S^{\frac{1}{2}\pi} \sin \tfrac{1}{2}V_m = -0{\cdot}0129.$$

(n.) *Formation of the Mean Sums of* $\cos \tfrac{3}{2}V_m$ *and* $\sin \tfrac{3}{2}V_m$.

These may be found with sufficient accuracy from the last Table (m) of Sums, part of which is given. In that table lines are drawn dividing the columns into three divisions of six each. These are treated in the way shown in the following example :—

H. and L.W. $\tfrac{3}{2}V_m$.

							Refer to preceding sorting (m).
I.	.	.	.	.	.	.	-1^{st} six of $2^{nd}-4^{th}$.
	.	.	.	−3	0	0	$+2^{nd}$ six of $1^{st}-3^{rd}$.
	−3	+1	.	.	.	.	$+3^{rd}$ six of $2^{nd}-4^{th}$.
$2^{nd}-4^{th}$	−3	+1	.	−3	0	0	
	.	.	.	.	.	.	$+1^{st}$ six of $1^{st}-3^{rd}$.
	.	+1	+1	+1	.	−2	$+2^{nd}$ six of $2^{nd}-4^{th}$.
	+1	−1	−2	.	.	.	-3^{rd} six of $1^{st}-3^{rd}$.
$1^{st}-3^{rd}$	+1	0	−1	+1	.	−2	
$2^{nd}-4^{th}$, rev.	0	0	−3	.	+1	−3	
II.	.	.	.	.	.	.	-1^{st} six of $2^{nd}-4^{th}$.
	.	.	.	.	.	−3	$+2^{nd}$ six of $1^{st}-3^{rd}$.
	+3	+3	.	.	.	.	$+3^{rd}$ six of $2^{nd}-4^{th}$.
$2^{nd}-4^{th}$	+3	+3	.	.	.	−3	
	.	.	.	.	.	.	$+1^{st}$ six of $1^{st}-3^{rd}$.
	.	.	.	.	−2	−3	$+2^{nd}$ six of $2^{nd}-4^{th}$.
	+2	−3	−2	.	.	.	-3^{rd} six of $1^{st}-3^{rd}$.
$1^{st}-3^{rd}$	+2	−3	−2	.	−2	−3	
$2^{nd}-5^{th}$, rev.	−3	.	.	.	+3	+3	
III.			&c.				&c.

We have now only 6 instead of 18 sub-divisions of the quadrant, but the cosine and sine numbers are found in exactly the same way as before.

The following example shows part of the treatment, and the cosine factors are those marked * in F.

Cosines.				Sums in pairs.				Differences.			Sum of 5 columns.	Cosine factors.	+	−
I.	II.		XIII.	I + II	III + IV.		XI + XII.	(I + II). − (III + IV).		(IX + X). − (XI + XII).				
+1	+5	&c.	0	+6	−2	&c.	+3	+8	&c.	0	+23	× 0·991	22·973	.
0	−3		−1	−3	+3		−1	−6		+1	− 8	0·924	.	7·392
+2	−2		0	0	0		−2	0		−4	−16	0·793	.	12·688
+1	.		+1	+1	−5	&c.	+1	+6	&c.	−1	+20	0·609	12·180	.
−1	−5	&c.	−1	−6	+2		−3	−8		+1	−17	0·383	.	6·511
+1	−6		+1	−5	−1		−3	−4		+7	+20	0·130	3·120	.
													38·273	26·591
													26·591	
													+11·682	
												Factor Ψ ...	× 0·00298	
												$S^{\circ} \cos \frac{3}{2} V_m =$	+0·0347	

The remaining process is exactly like that pursued before, and the four results are

$$S^{\circ}\cos\tfrac{3}{2}V_m = +0{\cdot}0347, \qquad S^{\circ}\sin\tfrac{3}{2}V_m = -0{\cdot}1830,$$

$$S^{\frac{1}{2}\pi}\cos\tfrac{3}{2}V_m = -0{\cdot}0479, \qquad S^{\frac{1}{2}\pi}\sin\tfrac{3}{2}V_m = -0{\cdot}0173.$$

(o.) *The Sorting of* $\frac{1}{2}V_m+V_s$ *and of* $\frac{1}{2}V_m-V_s$.

These angles have to be sorted without reference to the heights, or just as though all the heights were unity. Every entry is to be regarded as unity. The following example shows part of the sorting of $\frac{1}{2}V_m+V_s$, and 1 denotes a H.W., † a L.W.; by this device H. and L.W. may be sorted on the same paper.

We may also, if it is found convenient, put on it the sorting of $\frac{1}{2}V_m-V_s$ by adopting, say 0, to denote a H.W. and * a L.W., each one of these four signs denoting simply unity.

H. and L.W. $\frac{1}{2}V_m + V_s$. 1 for H.W., † for L.W.

1st quad.	Angles	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85
I.		1	.	†	.	.	1	.	†	†	.	.	.	1	1†	.	.	.	.
II.		†	.	.	1	†	.	.	1	.	†	.	1	.	.	1	.	.	†
				&c.						&c.						&c.			

2nd quad.	Angles	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175
I.		.	.	.	1†	.	.	.	.	.	.	1	.	.	.	1	†	.	.
II.		1	.	.	†	1	.	.	1†	.	.	.	†	.	.	.	†	1	.
					&c.					&c.						&c.			

3rd quad.	Angles 180, &c.
	&c.

4th quad.	Angles 270, &c.
	&c.

We then proceed to count these 1's and †'s just as was done with the number of entries in the sorting of $\frac{1}{2}V_m$, and to operate on them in the same way.

The results are

$$S^{o}\cos(\tfrac{1}{2}V_m - V_s) = -\cdot 0078, \qquad S^{o}\sin(\tfrac{1}{2}V_m - V_s) = -\cdot 0060,$$

$$S^{\frac{1}{2}\pi}\cos(\tfrac{1}{2}V_m - V_s) = +\cdot 0280, \qquad S^{\frac{1}{2}\pi}\sin(\tfrac{1}{2}V_m - V_s) = +\cdot 0078,$$

$$S^{o}\cos(\tfrac{1}{2}V_m + V_s) = +\cdot 1244, \qquad S^{o}\sin(\tfrac{1}{2}V_m + V_s) = +\cdot 0094,$$

$$S^{\frac{1}{2}\pi}\cos(\tfrac{1}{2}V_m + V_s) = +\cdot 0147, \qquad S^{\frac{1}{2}\pi}\sin(\tfrac{1}{2}V_m + V_s) = +\cdot 0834.$$

(p).—*Evaluation of* $F_m^{(o)}$, $G_m^{(o)}$, $f_m^{(o)}$, $g_m^{(o)}$, $F_m^{(\frac{1}{2}\pi)}$, $G_m^{(\frac{1}{2}\pi)}$, $f_m^{(\frac{1}{2}\pi)}$, $g_m^{(\frac{1}{2}\pi)}$, $F_s^{(o)}$, $G_s^{(o)}$, $f_s^{(o)}$, $g_s^{(o)}$, $F_s^{(\frac{1}{2}\pi)}$, $G_s^{(\frac{1}{2}\pi)}$, $f_s^{(\frac{1}{2}\pi)}$, $g_s^{(\frac{1}{2}\pi)}$.

These 16 coefficients are required to correct the four sums $S^{o}h \frac{\cos}{\sin}\frac{1}{2}V_m$, $S^{\frac{1}{2}\pi}h \frac{\cos}{\sin}\frac{1}{2}V_m$, for the influence of the tides M_2 and S_2.

I call $S^{o}h\cos\frac{1}{2}V_m$ + corrn., W, $S^{o}h\sin\frac{1}{2}V_m$ + corrn., X, and the other two Y and Z.

The correction to be applied to $S^{o}h\cos\frac{1}{2}V_m$ to get W is

$$-[F_m^{(o)}A_m + G_m^{(o)}B_m + F_s^{(o)}A_s + G_s^{(o)}B_s],$$

and the correction to be applied to $S^{o}h\sin\frac{1}{2}V_m$ to get X is

$$-[f_m^{(o)}A_m + g_m^{(o)}B_m + f_s^{(o)}A_s + g_s^{(o)}B_s],$$

and the two other corrections are given by symmetrical formulæ with $(\frac{1}{2}\pi)$ in place of (o).

These coefficients are computed from S^{o} and $S^{\frac{1}{2}\pi}$ of $\frac{\cos}{\sin}(\frac{1}{2}V_m \pm V_m)$ and of $\frac{\cos}{\sin}(\frac{1}{2}V_m \pm V_s)$, as given in (m) (n) (o). It must be especially noticed that we have above in (m) computed S^{o} and S of $\sin\frac{1}{2}V_m$; but $\frac{1}{2}V_m - V_m = -\frac{1}{2}V_m$, so that the signs of our previous results must be changed in these two cases.

If we remark that k_m and k_s are constants found by theoretical considerations, that A_m, B_m, A_s, B_s, are already found, and that in the first column we are compelled to omit the affixes to the letters S, k, and the F's and G's, because they indicate various sorts of S's and k's and F's and G's in the different columns, the computations in the following table are easily followed :—

	S°. $k_m = 2$. $V_p = V_m$.	S°. $k_s = 2{\cdot}07$. $V_p = V_s$.	$S^{\frac{1}{2}\pi}$. $k_m = 2$. $V_p = V_m$.	$S^{\frac{1}{2}\pi}$. $k_s = 2{\cdot}07$. $V_p = V_s$.
$\frac{1}{2}S\cos(\frac{1}{2}V_m - V_p)$	− ·0261	− ·0039	− ·0084	+ ·0140
$\frac{1}{2}S\cos(\frac{1}{2}V_m + V_p)$	+ ·0174	+ ·0622	+ ·0240	+ ·0074
Sum Σ	− ·0087	+ ·0583	− ·0324	+ ·0214
Diff. Δ	− ·0435	− ·0661	+ ·0156	+ ·0066
$k\Sigma$	− ·0174	+ ·1207	− ·0648	+ ·0443
$k\Delta$	− ·0870	− ·1368	+ ·0312	+ ·0137
$\Sigma + k\Delta = F$	− ·0957	− ·0785	− ·0012	+ ·0351
$\Delta + k\Sigma = g$	− ·0609	+ ·0546	− ·0492	+ ·0509
$\frac{1}{2}S\sin(\frac{1}{2}V_m - V_p)$	− ·0059	− ·0030	+ ·0065	+ ·0039
$\frac{1}{2}S\sin(\frac{1}{2}V_m + V_p)$	− ·0915	+ ·0047	− ·0087	+ ·0417
Sum σ	− ·0974	+ ·0017	− ·0022	+ ·0456
Diff. δ	+ ·0856	− ·0077	+ ·0152	− ·0378
$k\sigma$	− ·1948	+ ·0035	− ·0044	+ ·0944
$k\delta$	+ ·1712	− ·0159	+ ·0304	− ·0782
$\sigma + k\delta = f$	+ ·0738	− ·0142	+ ·0282	− ·0326
$-(\delta + k\sigma) = G$	+ ·1092	+ ·0042	− ·0108	− ·0566

Coefficients......	$-A_m$.	$-B_m$	$-A_s$.	$-B_s$.		
$(S^{\circ}h \cos \frac{1}{2}V_m) - 10 \cdot 50$	$(F_m{}^{o})$ − ·0957	$(G_m{}^{o})$ + ·1092	$(F_s{}^{o})$ − ·0785	$(G_s{}^{o})$ + ·0042	=	W.
$(S^{\circ}h \sin \frac{1}{2}V_m) + 8 \cdot 04$	$(f_m{}^{o})$ + ·0738	$(g_m{}^{o})$ − ·0609	$(f_s{}^{o})$ − ·0142	$(g_s{}^{o})$ ·0546	=	X.
$(S^{\frac{1}{2}\pi}h \cos \frac{1}{2}V_m) + 0 \cdot 40$	$(F_m{}^{\frac{1}{2}\pi})$ − ·0012	$(G_m{}^{\frac{1}{2}\pi})$ − ·0108	$(F_s{}^{\frac{1}{2}\pi})$ + ·0351	$(G_s{}^{\frac{1}{2}\pi})$ − ·0326	=	Y.
$(S^{\frac{1}{2}\pi}h \sin \frac{1}{2}V_m) + 3 \cdot 74$	$(f_m{}^{\frac{1}{2}\pi})$ + ·0282	$(g_m{}^{\frac{1}{2}\pi})$ − ·0492	$(f_s{}^{\frac{1}{2}\pi})$ − ·0566	$(g_s{}^{\frac{1}{2}\pi})$ + ·0509	=	Z.
Multiply by	$-A_m = +30 \cdot 58$,	$-B_m = -38 \cdot 47$,	$-A_s = -21 \cdot 08$,	$-B_s = -3 \cdot 53$.		
−10·50	−2·91	−4·23	+1·66	−0·01	=	W.
+ 8·04	+2·24	+2·36	+0·30	−0·19	=	X.
+ 0·40	−0·04	+0·42	−0·74	+0·12	=	Y.
+ 3·74	+0·86	+1·91	+1·20	−0·18	=	Z.

W.		X.		Y.		Z.	
+	−	+	−	+	−	+	−
	10·50	8·04		0·40		3·74	
	2·91	2·24			0·04	0·86	
	4 23	2·36		0·42		1·91	
1·66		0·30			0·74	1·20	
	0·01		0·19	0·12			0·18
1·66	17·65	12·94		0·94	0·78	7·71	
	1·66	0·19		0·78		0·18	
	−15·99	+12·75		+0·16		+7·53	

$$W = -15{\cdot}99 \qquad X = +12{\cdot}75$$
$$Z = +\ 7{\cdot}53 \qquad Y = +\ 0{\cdot}16$$
$$W+Z = -\ 8{\cdot}46 \qquad X+Y = +12{\cdot}91$$
$$W-Z = -23{\cdot}52 \qquad X-Y = +12{\cdot}59$$
$$\tfrac{1}{2}(W+Z) = -\ 4{\cdot}23 \qquad \tfrac{1}{2}(X+Y) = +\ 6{\cdot}46$$
$$\tfrac{1}{2}(W-Z) = -11{\cdot}76 \qquad \tfrac{1}{2}(X-Y) = +\ 6{\cdot}30$$

(q.) *Computation of Astronomical and other Constants.*

Find s_o, the moon's mean longitude (see 'Nautical Almanac,' Moon's Libration), and h_o the sun's mean longitude (sidereal time reduced to angle) from the 'Nautical Almanac,' and p_o the longitude of moon's perigee, from Baird's Manual,* Appendix Table XII (there called π), at the epoch 0^h, January 1, 1887, Bombay mean time, in E. Longitude $4^h{\cdot}855$.

From Baird, Tables XIV, XV, XVIII, find N the longitude of Moon's node, and I, ν, ξ at mid-period, February 14, 1887.†

With the value of I find f_m from XIX (1) for the tides M_2, N, L; from XIX (3) find f_o for the tide O; from XIX (8) find f' for the tide K_1; from XIX (9) find f'' for the tide K_2; from XX find ν' for the tide K_1; and from XXI find $2\nu''$ for the tide K_2.

The results are

$$s_o = 359^\circ{\cdot}43, \qquad h_o = 280^\circ{\cdot}63, \qquad p_o = 165^\circ{\cdot}36.$$
$$\nu = -9^\circ{\cdot}60, \qquad \xi = -9^\circ{\cdot}00.$$
$$1/f_m = 0{\cdot}9709, \qquad 1/f_o = 1{\cdot}161, \qquad f' = 0{\cdot}915, \qquad f'' = 0{\cdot}802.$$
$$\nu' = -6^\circ{\cdot}30, \qquad 2\nu'' = -11^\circ{\cdot}75.$$

Then compute initial equilibrium arguments, in the symbol for which the subscript letters indicate the tides referred to,—

$$u_m = 2(h_o-\nu)-2(s_o-\xi), \qquad u_o = (h_o-\nu)-2(s_o-\xi)+\tfrac{1}{2}\pi, \qquad u_s = 0^\circ,$$
$$= 203^\circ{\cdot}60, \qquad = 3^\circ{\cdot}37,$$
$$\text{for } K_1,\ u' = h_o-\nu'-\tfrac{1}{2}\pi, \qquad \text{for } K_2,\ u'' = 2h_o-2\nu'',$$
$$= 196^\circ{\cdot}93, \qquad = 213^\circ{\cdot}01,$$
$$u_n = u_m-(s_o-p_o), \qquad u_l = u_m+(s_o-p_o)+\pi,$$
$$= 9^\circ{\cdot}53, \qquad = 217^\circ{\cdot}67,$$
$$u_p = -h_o+\tfrac{1}{2}\pi = 169^\circ{\cdot}37.$$

* 'Manual for Tidal Observations,' by Major Baird. Taylor and Francis, Fleet Street, 1886.

† In making these reductions I have really used the value of N for July 1, 1887, because I am operating on tidal *predictions* made for the whole year 1887, which were doubtless made with mean N for that year. The difference is almost insensible.

We have already shown in B the way of computing Π, and $\Pi = 1\cdot034$.*

In C and D we have shown how to compute j, i, l, and $j = +6^\circ\cdot46$, $i = +6^\circ\cdot52$, $l = -0^\circ\cdot97$.

By the formula in B, with $\alpha_n = 86^\circ\cdot97$ for 6 semi-lunations,

$$\omega = 2h_0 - 2\nu'' + \alpha_n = u'' + \alpha_n$$

$$= 299^\circ\cdot97 = -60^\circ\cdot03.$$

By the formula in D, with $\beta_n = 87^\circ\cdot52$ for XIII quarter-lunar periods.

$$\theta = 2h_0 - \nu' + l + \beta_n,$$

$$= 294^\circ\cdot11 = -65^\circ\cdot89.$$

By the formula in B, viz. :—

$$U \cos\phi = \Pi + \lambda_n f'' \cos\omega,$$

$$U \sin\phi = \lambda_n f'' \sin\omega.$$

With $\log\lambda_n = 9\cdot2517$ for 6 semi-lunations, and with the above values of Π, f'', ω :—

$$\phi = -6^\circ\cdot40, \qquad (+)\log U = 0\cdot0464.$$

By the formula in D, viz :—

$$T \cos\psi = f' - \rho_n \cos\theta,$$

$$T \sin\psi = \rho_n \sin\theta,$$

with $\log\rho_n = 9\cdot4618$ for XIII quarter-lunar periods, and with the above values of f' and θ :—

$$\psi = -18^\circ\cdot35, \qquad (+)\log T = 9\cdot9241.$$

(r.) *Final Evaluation of* M_2.

From (j) $B_m = +38\cdot47$, $A_m = -30\cdot58$, $\tan\zeta_m = \frac{B_m}{A_m}$;

B_m is + and A_m is –, so that ζ_m lies in second quadrant; whence

$$\zeta_m = \pi - 51^\circ\cdot51 = 128^\circ\cdot49.$$

Then $$H_m = \frac{1}{f_m} . B_m \operatorname{cosec}\zeta_m;$$

* As the Indian tide predicting instrument takes no account of solar parallax, I should in reality have done better to take Π as unity. But of course this consideration does not apply to real observations.

whence, on reducing from inches to feet,

$$H_m = 3{\cdot}98 \text{ ft.}$$

Also $$\kappa_m = \zeta_m + u_m = 128^\circ{\cdot}49 + 203^\circ{\cdot}60 = 332^\circ{\cdot}09,$$

where the value of u_m is taken from (q).

(s.) *Final Evaluation of* N *and* L.

Taking the values of P, Q, R, S from (i),

$$f_m H_n \sin(\zeta_n - j) = -P = -10{\cdot}72, \quad f_m H_l \sin(\zeta_l + j) = +Q = +0{\cdot}66,$$
$$f_m H_n \cos(\zeta_n - j) = -S = +\ 7{\cdot}16, \quad f_m H_l \cos(\zeta_l + j) = -R = +1{\cdot}22.$$

$\zeta_n - j$ lies in 4th quad., $\zeta_l + j$ lies in 1st quad.;

whence $$\zeta_n - j = -56^\circ{\cdot}27.$$

Then $$H_n = \frac{1}{f_m} \operatorname{cosec}(\zeta_n - j) \times (-P);$$

whence, on reducing from inches to feet,

$$H_n = 1{\cdot}04 \text{ ft.}$$

Again, since from (q) $j = +6^\circ{\cdot}54$, we have $\zeta_n = -49^\circ{\cdot}73 = 310^\circ{\cdot}27$, and $\kappa_n = \zeta_n + u_n = 310^\circ{\cdot}27 + 9^\circ{\cdot}53 = 319^\circ{\cdot}80$, where the value of u_n is taken from (q).

Turning to the second pair of equations,

$$\zeta_l + j = 28^\circ{\cdot}4.$$

Then $$H_l = \frac{1}{f_m} \sec(\zeta_l + j) \times (-R);$$

whence, on reducing from inches to feet,

$$H_l = 0{\cdot}11 \text{ ft.}$$

Again, since $j = +6^\circ{\cdot}5$, we have $\zeta_l = 21^\circ{\cdot}9$, and $\kappa_l = \zeta_l + u_l = 21^\circ{\cdot}9 + 217^\circ{\cdot}7 = 239^\circ{\cdot}6$, where the value of u_l is taken from (q).

(t.) *Final Evaluation of* S_2 and K_2

From (k) $B_s = +3{\cdot}62, \quad A_s = +21{\cdot}08; \quad \tan\zeta_s = \dfrac{B_s}{A_s};$

B_s and A_s are +, so that ζ_s lies in 1st quadrant;

whence $$\zeta_s = 9^\circ{\cdot}71.$$

Then $$H_s = \frac{A_s \sec \zeta_s}{U},$$

whence, with log U already found in (q) as 0·0464, and, reducing inches to feet,

$$H_s = 1{\cdot}60 \text{ ft.}$$

Again $$\kappa_s = \zeta_s + \phi = 9^\circ{\cdot}71 - 6^\circ{\cdot}40 = 3^\circ{\cdot}31,$$

where the value of ϕ is taken from (q).

Lastly,

$$H'' = 0{\cdot}272 H_s = 0{\cdot}44 \text{ ft.}, \qquad \text{and } \kappa'' = \kappa_s = 3^\circ.$$

The factor 0·272 is an absolute constant.

(u.) *Final Evaluation of* K_1, O, P.

Taking the values of $\frac{1}{2}(W-Z)$, $\frac{1}{2}(X+Y)$ from (p),

$$TH' \sin(\zeta'+i-\psi) = \tfrac{1}{2}(W-Z) = -11{\cdot}76,$$

$$TH' \cos(\zeta'+i-\psi) = -\tfrac{1}{2}(X+Y) = -6{\cdot}47;$$

$\zeta'+i-\psi$ lies in third quadrant, and

$$\zeta'+i-\psi = \pi + 61^\circ{\cdot}2 = 241^\circ{\cdot}2.$$

Then since, from (q), $\psi = -18^\circ{\cdot}35$, we have $\zeta'+i = 222^\circ{\cdot}8$; and since from (q) $i = 6^\circ{\cdot}52$, therefore $\zeta' = 216^\circ{\cdot}3$; whence

$$\kappa' = \zeta' + u' = 216^\circ{\cdot}3 + 196^\circ{\cdot}9 = 53^\circ{\cdot}2,$$

where the value of u' is taken from (q).

Then $$H' = \frac{\frac{1}{2}(W-Z)}{T} \operatorname{cosec}(\zeta'+i-\psi),$$

whence, with log T already found in (q) as 9·9241, and reducing from inches to feet,

$$H' = 1{\cdot}33 \text{ ft.}$$

Also $$H_p = 0{\cdot}331 H' = 0{\cdot}44, \qquad \text{and } \kappa_p = \kappa' = 53^\circ.$$

The factor 0·331 is an absolute constant.

We now have to compute

$$L = \tfrac{1}{2}(X+Y) \tan \tfrac{1}{4}\epsilon + f'H' \cos(\zeta'+i) \tan \tfrac{1}{4}\epsilon,$$

$$M = \tfrac{1}{2}(W-Z) \tan \tfrac{1}{4}\epsilon - f'H' \sin(\zeta'+i) \tan \tfrac{1}{4}\epsilon,$$

where $\log \tan \frac{1}{4}\epsilon = 9{\cdot}0677$, an absolute constant for all times and places. With the values of f′ and $\frac{1}{2}(X+Y)$ and $\frac{1}{2}(W-Z)$ given above in (q) and (p), and with the values of H′ and $\zeta' + i$ just found, there results—

$$L = -0{\cdot}495, \qquad M = -0{\cdot}214.$$

Now
$$f_o H_o \sin(\zeta_o - i) = -\tfrac{1}{2}(W+Z) + L,$$
$$f_o H_o \cos(\zeta_o - i) = \tfrac{1}{2}(X-Y) + M.$$

We have found in (p)

$$\tfrac{1}{2}(W+Z) = -4{\cdot}23, \qquad \tfrac{1}{2}(X-Y) = +6{\cdot}30,$$

so that

$$f_o H_o \sin(\zeta_o - i) = +3{\cdot}73, \qquad f_o H_o \cos(\zeta_o - i) = +6{\cdot}09.$$

Whence $\zeta_o - i$ lies in the first quadrant, and

$$\zeta_o - i = 31^\circ{\cdot}50.$$

Then
$$H_o = \frac{1}{f_o}\left[\tfrac{1}{2}(X-Y) + M\right] \sec(\zeta_o - i),$$

whence, reducing from inches to feet,

$$H_o = 0{\cdot}69 \text{ ft.}$$

Again,

$$\kappa_o = \zeta_o + u_o = (\zeta_o - i) + i + u_o = 31^\circ{\cdot}50 + 6^\circ{\cdot}52 + 3^\circ{\cdot}37 = 41^\circ{\cdot}39,$$

where the value of u_o is taken from (q).

(v.) *Final Reduction of Mean Water Mark.*

We subtracted 99 inches from all the heights before using them, and the mean of the heights was then + 3·51 inches. Hence mean water is 102·51 inches, or 8·54 feet above the datum of the original tidal observations.

(w.) *Results of Reduction.*

		Mean of 9 yrs. obs.	Error of present calc. in inches and minutes.
Mean water, 8·54 ft.		8·223	4 in.
M_2	H = 3·98 ft.	4·043	$\frac{3}{4}$ in. too small.
	κ = 332°	330°	4^m too fast.
S_2	H = 1·60 ft.	1·625	$\frac{1}{3}$ in. too small.
	κ = 3°	3°	Nil.
K_2	H = 0·44 ft.	0·405	$\frac{1}{2}$ in. too large.
	κ = 3°	352°	22^m too fast.
N	H = 1·04 ft.	0·997	$\frac{1}{2}$ in. too large.
	κ = 320°	313°	11^m too fast.
L	H = 0·11 ft.	0·088	$\frac{1}{4}$ in. too large.
	κ = 240°	308°	$2^h\ 18^m$ too slow.
K_1	H = 1·33 ft.	1·396	$\frac{2}{3}$ in. too small.
	κ = 53°	45°	32^m too fast.
O	H = 0·69 ft.	0·658	$\frac{1}{3}$ in. too large.
	κ = 41°	48°	29^m too slow.
P	H = 0·44 ft.	0·404	$\frac{1}{2}$ in. too large.
	κ = 53°	43°	40^m too fast.

The second column is given because, if the calculation had been conducted by rigorous methods instead of approximately, my results should have agreed very nearly* with these. The causes of several of the discrepancies are explicable. The error of mean water mark is due to the necessity for neglecting the annual and semi-annual tides in a short series of observations. The error in the height of S_2 is partly due to my taking $\Pi = 1{\cdot}034$ instead of putting it equal to 1, as is virtually done in the Indian tidal instrument. If I had taken $\Pi = 1$, I should have had $H_s = 1{\cdot}65$ nearly. The error in phase in K_2 is a necessary incident of the shortness of the series of observations. The error of N must be due to the shortness of the series, which has not permitted an adequate elimination of the evectional and variational tides. The tide L is only about an inch in height, and accuracy of result could not be expected.

The magnitude of the error in time in the diurnal tides is disappointing, but it is clear that the length of observation has not been sufficient to disentangle the O tide from the K_1 tide. It may be remarked also that an error of 1° in phase makes twice as much differ-

* I do not know the exact values of the constants used in the Bombay Tide Table, which has been used as representing observation.

ence in time with the diurnal tides as with the semi-diurnal. The errors of P fall under the same category as those of K_2.

Lastly it is probable that all these errors would have been sensibly diminished if I had subtracted 103 inches from the heights all through instead of 99, and I know that this is to some extent the case.

(x.) *Verification.*

In a calculation of this kind some gross error of principle may have been committed, such, for example, as imputing to some of the κ's a wrong sign; and this is the kind of mistake which is easily overlooked in a mere verification of arithmetical processes. It is well, therefore, to test whether the tide heights and times are actually given by the computed constants. This is conveniently done by selecting some three or four tides from amongst those from which the reductions have been made, and it makes the calculation much shorter if we pick out cases in which it is H. or L.W. within a few minutes of noon.

For example, in the present case it was L.W. on February 16 (day 46) at $0^h\ 7^m$ P.M., and the height was 4 ft. 0 in.; again, it was H.W. on March 25 (day 83) at $0^h\ 3^m$ P.M., and the height was 13 ft. 3 in.

Now, if U denotes the value of any equilibrium argument whose value at the epoch, 0^h, January 1, was denoted in (q) by u, and if A_0 denotes the height of mean sea-level above datum, the expression for the height of water is :—

$$\begin{aligned} h = A_0 &+ f_m H_m \cos(U_m - \kappa_m) + H_s \cos(U_s - \kappa_s) + f'' H'' \cos(U'' - \kappa'') \\ &+ f_m H_n \cos(U_n - \kappa_n) + f_m H_l \cos(U_l - \kappa_l) + f' H' \cos(U' - \kappa') \\ &+ f_o H_o \cos(U_o - \kappa_o) + H_p \cos(U_p - \kappa_p). \end{aligned}$$

The time of H.W. depends on a formula involving the sines of the same angles in place of cosines.

Since we have chosen cases where it is H. or L.W. at noon, the U's exceed the u's by an exact number of days' motion.

The evaluation of the separate terms may be conveniently made by means of an ordinary nautical traverse table, where (neglecting the decimal point) fH is represented by the "Distance," and fH $\cos(U - \kappa)$ is given by "Latitude," and fH $\sin(U - \kappa)$ by "Departure."

If we know the time of H. or L.W. within 20^m or so, the following calculation will give the true time and height. The computation applies to the first of the two cases where we know that there should be a L.W. at about 0^h of day 46. The increments of argument are computed from the Table G, and the κ's are subtracted either by actual subtraction or by addition of $2\pi - \kappa$.

	M_2.	S_2.	K_2.	N.	L.	K_1.	O.	P.
Increment in 40^d	$464°·7$		$78°·9$	$-57°·9$	$-452·7$	$39·4$	$425·3$	
Ditto 6^d	$-146·3$		$11·8$	$-224·7$	$-67·9$	$5·9$	$-152·2$	(see K_1)
Ditto 46^d	$318·4$		$90·7$	$-282·6$	$-520·6$	$45·3$	$273·1$	$-45·3$
$u =$	$203·6$		$213·0$	$9·5$	$217·7$	$196·9$	$3·4$	$169·4$
$U =$	$522·0$		$303·7$	$-273·1$	$-302·9$	$242·2$	$276·5$	$124·1$
$-\kappa =$	$-332·1$	$-3·3$	$-3·3$	$+40·2$	$+120·4$	$-53·1$	$-41·4$	$-53·1$
$U-\kappa =$	$189·9$	$-3·3$	$300·4$	$-232·9$	$-182·5$	$189·1$	$235·1$	$71·0$
$U-\kappa =$	$\pi+10$	-3	-60	$\pi-53$	$\pi-3$	$\pi+9$	$\pi+55$	71

		f.	H.	fH.	$U-\kappa$.	fH cos $(U-\kappa)$ +	fH cos $(U-\kappa)$ −	fH sin $(U-\kappa)$ +	fH sin $(U-\kappa)$ −
Semidiurnals.	M_2	1·03	3·98	4·10	$\pi+10°$		4·04		0·71
	S_2	1·0	1·60	1·60	-3	1·60			0·08
	K_2	0·80	0·44	0·35	-60	0·18			0·30
	N	1·03	1·04	1·07	$\pi-53$		0·64	0·86	
	L	1·03	0·11	0·11	$\pi-3$		0·11	0·01	
						+1·78	−4·79	+0·87	−1·09
							+1·78		+0·87
						$A_2 =$	−3·01	$B_2 =$	−0·22
Diurnals.	K_1	0·915	1·33	1·22	$\pi-9$		1·21		0·19
	O	0·86	0·69	0·59	$\pi-55$		0·34		0·48
	P	1·00	0·44	0·44	71	0·14		0·42	
						+0·14	−1·55	+0·42	−0·67
							+0·14		+0·42
						$A_1 =$	−1·41	$B_1 =$	−0·25
						$\tfrac{1}{4}A_1 =$	−0·35	$\tfrac{1}{2}B_1 =$	−0·13
						$A_2+\tfrac{1}{4}A_1 =$	−3·36	$B_2+\tfrac{1}{2}B_1 =$	−0·35

$$\text{Time} = -120^m\left(\frac{B_2+\tfrac{1}{2}B_1}{A_2+\tfrac{1}{4}A_1}\right) = -120^m \times \frac{35}{336} = -12^m = 11^h\ 48^m \text{ A.M.}$$

Tabular time = $0^h\ 7^m$ P.M.
Error = -19^m.

$A_2 + A_1 = -4·42$
Mean water $+8·54$

Height = $4·12$

Height L.W.	4 ft. 1 in.
Tabular height	4 0
Error	+1

In the second case referred to above, the calculated height is 13 ft. 6 in., and the tabulated height 13 ft. 3 in., and the error in time is -15^m.

These results are as good as might be expected, and, considered as a prediction, would be amply sufficient for navigational purposes.