III. "On Jacobi's Figure of Equilibrium for a Rotating Mass of Fluid." By G. H. DARWIN, M.A., LL.D., F.R.S., Fellow of Trinity College and Plumian Professor in the University of Cambridge. Received October 12, 1886.

I am not aware that any numerical values have ever been determined for the axes of the ellipsoids, which are figures of equilibrium of a rotating mass of fluid.*

In the following paper the problem is treated from the point of view necessary for reducing the formulæ to a condition for computation, and a table of numerical results is added.

Let a, b, c be the semi-axes of a homogeneous ellipsoid of unit density; let the origin be at the centre and the axes of x, y, z be in the directions a, b, c.

Then if we put—

$$A^2=a^2+u, \qquad B^2=b^2+u, \qquad C^2=c^2+u, \text{ and}$$

$$\Psi=\int_0^\infty \frac{du}{ABC}, \quad \text{. . . . . (1)}$$

it is known† that the potential of the ellipsoid at an internal point x, y, z is given by—

$$V=\pi abc\left[\Psi+\frac{x^2}{a}\frac{d\Psi}{da}+\frac{y^2}{b}\frac{d\Psi}{db}+\frac{z^2}{c}\frac{d\Psi}{dc}\right] \quad \text{. . . . . (2)}$$

Now let us introduce a new notation, and let

* The following list of papers bearing on this subject is principally taken from a report to the British Association, 1882, by W. M. Hicks:—

Jacobi, 'Acad. des Sciences,' 1834; Liouville, 'Journ. École Polytech.,' vol. xiv, p. 289; Ivory, 'Phil. Trans.,' 1838, Pt. I, p. 57; Pontécoulant, 'Syst. du Monde,' vol. ii. The preceding are proofs of the theorem, and in more detail we have:—C. O. Meyer, 'Crelle,' vol. xxiv, p. 44; Liouville, 'Liouville's Journ.,' vol. xvi, p. 241; a remarkable paper by Dirichlet and Dedekind, 'Borchardt's Journ.,' vol. lviii, pp. 181 and 217; Riemann, 'Abh. K. Ges. Wiss. Göttingen,' vol. ix, 1860, p. 3; Brioschi, 'Borchardt's Journ.,' vol. lix, p. 63; Padova, 'Ann. della Sc. Norm. Pisa,' 1868–9 (being Dirichlet and Riemann's work with additions); Greenhill, 'Proc. Camb. Phil. Soc.,' vol. iii, p. 233 and vol. iv, p. 4; Lipschitz, 'Borch. Journ.,' vol. lxxviii, p. 245; Hagen, 'Schlömilch Zeitsch. Math.,' vol. xxiv, p. 104; Betti, 'Ann. di Matem.' vol. x, p. 173 (1881); Thomson and Tait's 'Nat. Phil.' (1883), Part II, §778; a very important paper by Poincaré, 'Acta Mathem.,' 7, 3 and 4 (1885).

† Thomson and Tait's 'Nat. Phil.' (1883) §494, *l*. The form in which the formula is here given is slightly different from that in (8), (11), (15) of §§ 494, *k*, *l*.

$$\left.\begin{aligned} &c=a\cos\gamma, \quad \sin\alpha=\sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \text{and } b=a\cos\beta, \\ &\text{so that } \sin\beta=\sin\alpha\sin\gamma, \text{ and } b=a\sqrt{(1-\sin^2\alpha\sin^2\gamma.)} \end{aligned}\right\} \quad . \quad (3)$$

$$\left.\begin{aligned} &\text{Also let} \quad A^2=u+a^2=\frac{a^2-c^2}{\sin^2\theta}=a^2\frac{\sin^2\gamma}{\sin^2\theta}, \\ &\text{whence} \quad B^2=u+b^2=\frac{a^2\sin^2\gamma}{\sin^2\theta}(1-\sin^2\alpha\sin^2\theta), \\ &\qquad C^2=u+c^2=\frac{a^2\sin^2\gamma}{\sin^2\theta}\cos^2\theta, \\ &\text{and} \quad du=-\frac{2a^2\sin^2\gamma}{\sin^3\theta}\cos\theta\, d\theta, \\ &\text{and} \quad \int_0^\infty du=2a^2\sin^2\gamma\int_0^\gamma\frac{\cos\theta}{\sin^3\theta}d\theta=2a^2\sin^2\gamma\int_0^\gamma\frac{\cos\gamma}{\sin^3\gamma}d\gamma. \end{aligned}\right\} \quad . \quad (4)$$

Lastly, let
$$\Delta=\sqrt{(1-\sin^2\alpha\sin^2\gamma)},$$
and in accordance with the usual notation of elliptic integrals let

$$\mathrm{F}=\int_0^\gamma\frac{d\gamma}{\Delta}, \qquad \mathrm{E}=\int_0^\gamma\Delta\, d\gamma. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then we have the following transformations:—

$$\left.\begin{aligned} \Psi&=\int_0^\infty\frac{du}{ABC}=\frac{2}{a\sin\gamma}\mathrm{F} \\ -\frac{d\Psi}{ada}&=\int_0^\infty\frac{du}{A^3BC}=\frac{2}{a^3\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta}d\gamma=\frac{2}{a^3\sin^2\alpha\sin^3\gamma}(\mathrm{F}-\mathrm{E}) \\ -\frac{d\Psi}{bdb}&=\int_0^\infty\frac{du}{AB^3C}=\frac{2}{a^3\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta^3}d\gamma \\ -\frac{d\Psi}{cdc}&=\int_0^\infty\frac{du}{ABC^3}=\frac{2}{a^3\sin^3\gamma}\int_0^\gamma\frac{\tan^2\gamma}{\Delta}d\gamma \end{aligned}\right\} \quad . \quad (6)$$

It remains to reduce the last two of (6) to elliptic integrals.

If k and k' be the modulus and its complement, the following are known transformations in the theory of elliptic functions, viz.:—

$$\int_0^\gamma\frac{d\gamma}{\Delta^3}=\frac{1}{k'^2}\mathrm{E}-\frac{k^2\sin\gamma\cos\gamma}{k'^2\Delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\int_0^\gamma\frac{\tan^2\gamma}{\Delta}d\gamma=\frac{\Delta\tan\gamma}{k'^2}-\frac{1}{k'^2}\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence
$$\int_0^\gamma\frac{\sin^2\gamma}{\Delta^3}d\gamma=\frac{1}{k^2}\int_0^\gamma\frac{1-\Delta^2}{\Delta^3}d\gamma=\frac{1}{k^2k'^2}\mathrm{E}-\frac{\sin\gamma\cos\gamma}{k'^2\Delta}-\frac{1}{k^2}\mathrm{F}. \quad . \quad (9)$$

In the present case $k=\sin\alpha$, $k'=\cos\alpha$, $\Delta=\cos\beta$. Thus (8) and (9) enable us to complete the required transformation to elliptic integrals of (6).

Substituting then from (6) (8) (9) in the expression

$$V=\tfrac{3}{4}m\left\{\Psi+\frac{x^2}{a}\frac{d\Psi}{da}+\frac{y^2}{b}\frac{d\Psi}{db}+\frac{z^2}{c}\frac{d\Psi}{dc}\right\},$$

where $m=\tfrac{4}{3}\pi abc=\tfrac{4}{3}\pi a^3\cos\beta\cos\gamma$, we have

$$V\div\tfrac{3}{4}m=\frac{2}{a\sin\gamma}F+\frac{2}{a^3\sin^3\gamma}\left[\frac{x^2}{\sin^2\alpha}(E-F)+y^2\left(\frac{\sin\gamma\cos\gamma}{\cos^2\alpha\cos\beta}+\frac{F}{\sin^2\alpha}-\frac{E}{\sin^2\alpha\cos^2\alpha}\right)+\frac{z^2}{\cos^2\alpha}(E-\tan\gamma\cos\beta)\right]. \quad (10)$$

Now suppose the ellipsoid to be rotating about the axis of z with an angular velocity ω, and let us choose the axes a, $a\cos\beta$, $a\cos\gamma$, and the angular velocity ω, so that the surface may be a surface of equilibrium.

For this purpose $V+\tfrac{1}{2}\omega^2(x^2+y^2)=$constant, must be identical with

$$\frac{x^2}{a^2}+\frac{y^2}{a^2\cos^2\beta}+\frac{z^2}{a^2\cos^2\gamma}=1.$$

Now in (10) we have V in the form

$$V=Lx^2+My^2+Nz^2+P, \quad \ldots\ldots \quad (11)$$

whence $a^2(L+\tfrac{1}{2}\omega^2)=a^2(M+\tfrac{1}{2}\omega^2)\cos^2\beta=a^2N\cos^2\gamma.$

Hence

$$\left.\begin{aligned}&L-M+N\cos^2\gamma\tan^2\beta=0,\\&\tfrac{1}{2}\omega^2=N\cos^2\gamma-L,\\ \text{or}\quad&\tfrac{1}{2}\omega^2\sin^2\beta=M\cos^2\beta-L.\end{aligned}\right\}\quad\ldots\ldots\quad(12)$$

There are two kinds of solutions of these equations (12).

First, since

$$\left.\begin{aligned}L&=\pi bc\frac{d\Psi}{da}=-\pi a^3\cos\beta\cos\gamma\cdot\frac{2}{a^3\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta}d\gamma\\&=-2\pi\frac{\cos\beta\cos\gamma}{\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta}d\gamma,\\M&=\pi ac\frac{d\Psi}{db}=-\pi a^3\cos\beta\cos\gamma\cdot\frac{2}{a^3\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta^3}d\gamma\\&=-2\pi\frac{\cos\beta\cos\gamma}{\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta^3}d\gamma,\end{aligned}\right\}.\quad(13)$$

it is obvious that $L-M$ vanishes when $\alpha=0$.

And since when α vanishes, β also vanishes, the equation

$$L-M+N\cos^2\gamma\tan^2\beta=0$$

is satisfied by $\alpha=0, \quad \beta=0.$

That is to say there is a solution of the problem which makes $a=b$.

Thus there is solution which gives us an ellipsoid of revolution.

When $\sin\alpha=0$, we have also $\beta=0$, $\Delta=1$, and

$$L=-\frac{2\pi\cos\gamma}{\sin^3\gamma}\int_0^\gamma \sin^2\gamma\, d\gamma=\frac{\pi\cos\gamma}{\sin^3\gamma}(\sin\gamma\cos\gamma-\gamma),$$

$$N=-\frac{2\pi\cos\gamma}{\sin^3\gamma}\int_0^\gamma \tan^2\gamma\, d\gamma=\frac{\pi\cos\gamma}{\sin^3\gamma}(2\gamma-2\tan\gamma).$$

Therefore

$$\tfrac{1}{2}\omega^2=N\cos^2\gamma-L,$$

$$=\frac{\pi}{\tan^3\gamma}[2\gamma-2\tan\gamma-(1+\tan^2\gamma)(\sin\gamma\cos\gamma-\gamma)],$$

$$=\frac{\pi}{\tan^3\gamma}[\gamma(3+\tan^2\gamma)-3\tan\gamma], \quad \text{. . . . . (14)*}$$

and the eccentricity of the ellipsoid of revolution is $\sin\gamma$.

To find the other solution when α is not zero, we have by comparison between (10) and (11),

$$\left.\begin{aligned} L\,.\,\frac{\sin^2\alpha\sin^3\gamma}{2\pi\cos\beta\cos\gamma}&=\mathrm{E}-\mathrm{F},\\ M\,.\,\frac{\sin^2\alpha\sin^3\gamma}{2\pi\cos\beta\cos\gamma}&=\frac{\sin^2\alpha\sin\gamma\cos\gamma}{\cos^2\alpha\cos\beta}+\mathrm{F}-\mathrm{E}\sec^2\alpha,\\ N\,.\,\frac{\sin^2\alpha\sin^3\gamma}{2\pi\cos\beta\cos\gamma}&=\tan^2\alpha(\mathrm{E}-\tan\gamma\cos\beta).\end{aligned}\right\} \quad (15)$$

Hence the first of (12) gives

$$-(2\mathrm{F}-\mathrm{E})+\mathrm{E}\sec^2\alpha + \tan^2\alpha\tan^2\beta\cos^2\gamma(\mathrm{E}-\tan\gamma\cos\beta)-\frac{\sin^2\alpha\sin\gamma\cos\gamma}{\cos^2\alpha\cos\beta}=0,$$

* Compare with Thomson and Tait's 'Nat. Phil.,' § 771, (3); or any other work which gives a solution of the problem.

or

$$\mathrm{E}\sec^2\alpha[1+(\sin\alpha\tan\beta\cos\gamma)^2]-(2\mathrm{F}-\mathrm{E})$$
$$-\sec^2\alpha\sin\alpha\tan\beta\cos\gamma(1+\sin^2\beta)=0. \quad . \quad . \quad . \quad (16)$$

In order to adapt this for computation, we may introduce the auxiliary angles defined by

$$\tan\zeta=\sin\alpha\tan\beta\cos\gamma,\ \tan\delta=\sin\beta, \quad . \quad . \quad . \quad . \quad (17)$$

and the equation becomes

$$\mathrm{E}\sec^2\alpha\sec^2\zeta-(2\mathrm{F}-\mathrm{E})-\sec^2\alpha\sec^2\delta\tan\zeta=0. \quad . \quad . \quad (18)$$

The second of (12) gives

$$\frac{\omega^2}{4\pi}\cdot\frac{\sin^2\alpha\sin^3\gamma}{\cos\beta\cos\gamma}=\tan^2\alpha\cos^2\gamma(\mathrm{E}-\tan\gamma\cos\beta)-(\mathrm{E}-\mathrm{F}),$$

whence $$\frac{\omega^2}{4\pi\cos\beta\cos\gamma}=\frac{\mathrm{F}-\mathrm{E}}{\sin^2\alpha\sin^3\gamma}+\frac{\mathrm{E}\cos^2\gamma}{\sin^3\gamma\cos^2\alpha}-\frac{\cos\beta\cos\gamma}{\cos^2\alpha\sin^2\gamma},$$

$$\frac{\omega^2}{4\pi}=\cot\beta\operatorname{cosec}\beta\cot\gamma(\mathrm{F}-\mathrm{E})+\cot^3\gamma\cos\beta\sec^2\alpha\mathrm{E}-\cos^2\beta\cot^2\gamma\sec^2\alpha. \quad . \quad . \quad . \quad (19)$$

Some of the subsequent computations were, however, actually made from a formula deduced from the third of (12), which leads to

$$\frac{\omega^2}{4\pi}=\cot\beta\cot\gamma\operatorname{cosec}^3\beta(1+\cos^2\beta)(\mathrm{F}-\mathrm{E})-\cot^3\beta\cot\gamma\tan^2\alpha\operatorname{cosec}\beta\,\mathrm{E}$$
$$+\cot^2\beta\cot^2\gamma\sec^2\alpha. \quad . \quad . \quad (20)$$

By subtracting (20) from (19) we can deduce (16); hence it follows that (19) and (20) lead to identical results. Most of the subsequent results were computed from both (19) and (20), thus verifying the solution of (18).

The formulæ (18) and (19) are suitable for finding the solution, except when α is small or nearly 90°, when the elliptic integrals become awkward to use. I have, therefore, found approximate formulæ for these cases, but as the algebraic process necessary to establish them is somewhat tedious, the details are given in a note.*

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* *Approximate Solutions of the Problem.*

From (7) we have

$$\int_0^\gamma\frac{\sin^4\gamma}{\Delta^3}d\gamma=\frac{1}{k^4}\int_0^\gamma\frac{(1-\Delta^2)^2}{\Delta^3}d\gamma$$
$$=\frac{1}{k^4k'^2}\mathrm{E}-\frac{1}{k^4}(2\mathrm{F}-\mathrm{E})-\frac{\sin\gamma\cos\gamma}{k^2k'^2\Delta}. \quad . \quad . \quad . \quad . \quad (a)$$

Approximate Solutions of the Problem—continued.

Now, since (16) may be written

$$\frac{1}{k'^2}\mathrm{E}-(2\mathrm{F}-\mathrm{E})-\frac{k^2\sin\gamma\cos\gamma}{k'^2\Delta}=\frac{k^4}{\Delta^2}\sin^2\gamma\cos^2\gamma\left[-\frac{1}{k'^2}\mathrm{E}+\frac{\Delta}{k'^2}\tan\gamma\right],$$

it follows from (a) and (8) that it may be written

$$\Delta^2\int_0^\gamma\frac{\sin^4\gamma}{\Delta^3}\,d\gamma=\sin^2\gamma\cos^2\gamma\int_0^\gamma\frac{\tan^2\gamma}{\Delta}d\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Again, we may write (19) thus:—

$$\frac{\omega^2}{4\pi}=\frac{\Delta\cos\gamma}{k^2\sin^3\gamma}(\mathrm{F}-\mathrm{E})+\frac{\Delta\cos^3\gamma}{k'^2\sin^3\gamma}\mathrm{E}-\frac{\Delta^2\cos^2\gamma}{k'^2\sin^2\gamma}$$

$$=\frac{\Delta\cos\gamma}{\sin^3\gamma}\int_0^\gamma\frac{\sin^2\gamma}{\Delta}d\gamma-\frac{\Delta\cos^3\gamma}{\sin^3\gamma}\int_0^\gamma\frac{\tan^2\gamma}{\Delta}d\gamma$$

$$=\frac{\Delta\cos\gamma}{\sin^3\gamma}\left[\int_0^\gamma\frac{\sin^2\gamma}{\Delta}d\gamma-\cos^2\gamma\int_0^\gamma\frac{\tan^2\gamma}{\Delta}d\gamma\right]. \quad . \quad . \quad . \quad . \qquad (c)$$

It is easier to develop the equations when written in the forms (b) and (c) than when we work directly from the elliptic integrals.

Write for brevity

$k=\sin\alpha$, $g=\cos\alpha$, $p=\cos\gamma$, $q=\sin\gamma$, $\mathrm{Q}=\tan\gamma$, $\Lambda=\log_e\cot(\frac{1}{4}\pi-\frac{1}{2}\gamma)$.

First, when k is small—

The following definite integrals are required:—

$$\int q^n d\gamma=-\frac{1}{n}q^{n-1}p+\frac{n-1}{n}\int q^{n-2}d\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

$$\int\frac{q^n}{p^2}d\gamma=\frac{q^{n-1}}{p}-(n-1)\int q^{n-2}d\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

From (d) and from $\dfrac{1}{\Delta^3}=\dfrac{1}{(1-k^2q^2)^{\frac{3}{2}}}=1+\dfrac{3}{2}k^2q^2+\dfrac{3.5}{2.4}k^4q^4+\ldots$

$$\int\frac{q^4}{\Delta^3}d\gamma=-\frac{1}{4}q^3p-\frac{3}{4.2}qp+\frac{3.1}{4.2}\gamma+\frac{3}{2}k^2\left(-\frac{1}{6}q^5p-\frac{5}{6.4}q^3p-\frac{5.3}{6.4.2}qp+\frac{5.3.1}{6.4.2}\gamma\right)+. \quad (f)$$

$\Delta^2=1-k^2q^2$, and

$$\Delta^2\int\frac{q^4}{\Delta^3}d\gamma=-\frac{1}{4}q^3p-\frac{3}{4.2}qp+\frac{3.1}{4.2}\gamma+k^2\left[-\left(\frac{3.1}{6.2}-\frac{1}{4}\right)q^5p-\left(\frac{3.5}{6.4.2}-\frac{3.1}{4.2}\right)q^3p\right.$$
$$\left.-\frac{3.5.3}{2.6.4.2}qp+\gamma\left(\frac{3.5.3.1}{2.6.4.2}-\frac{3.1}{4.2}q^2\right)\right]+\ldots$$

$$=-\frac{1}{4}q^3p-\frac{3}{8}qp+\frac{3}{8}\gamma+k^2\left[\frac{1}{16}q^3p-\frac{15}{32}qp+\gamma\left(\frac{15}{32}-\frac{3}{8}q^2\right)\right]. \quad . \quad . \quad . \qquad (g)$$

Again, $\dfrac{1}{\Delta}=1+\dfrac{1}{2}k^2q^2+\ldots$, and by ($e$)—

Approximate Solutions of the Problem—continued.

$$\int \frac{Q^2}{\Delta} d\gamma = \int \frac{q^2}{p^2 \Delta} d\gamma = \frac{q}{p} - \gamma + \frac{1}{2} k^2 \left[\frac{q^3}{p} + \frac{3}{2} qp - \frac{3}{2} \gamma \right] + \dots$$

$$p^2 q^2 \int \frac{Q^2}{\Delta} d\gamma = pq^3 - p^2 q^2 . \gamma + \frac{1}{2} k^2 \left[pq^5 - \frac{3}{2} pq^5 + \frac{3}{2} pq^3 - \frac{3}{2} \gamma p^2 q^2 \right] + \dots$$

$$= pq^3 - p^2 q^2 . \gamma + k^2 \left(-\frac{1}{4} pq^5 + \frac{3}{4} pq^3 - \frac{3}{4} \gamma p^2 q^2 \right) + \dots \qquad (h)$$

The equation (b) is $p^2 q^2 \int \frac{Q^2}{\Delta} d\gamma = \Delta^2 \int \frac{q^4}{\Delta^3}$, whence, equating ($g$) and ($h$),

$$\frac{5}{4} q^3 p + \frac{3}{8} pq - \frac{3}{8} \gamma \left(1 + \frac{8}{3} p^2 q^2 \right) + k^2 \left[-\tfrac{1}{4} pq^5 + \frac{11}{16} pq^3 + \frac{15}{32} pq - \gamma \left(\frac{15}{32} + \frac{3}{8} q^2 - \frac{3}{4} q^4 \right) \right] + \dots = 0.$$

If k be zero, we have $\quad \frac{5}{4} q^3 p + \frac{3}{8} pq - \gamma \left(\frac{3}{8} + p^2 q^2 \right) = 0.$

This easily reduces to $\quad \gamma = \frac{\sin 2\gamma - \frac{5}{16} \sin 4\gamma}{1 - \frac{1}{4} \cos 4\gamma}, \qquad (i)$

of which the solution is $\gamma = 54° \; 21' \; 27''$, as stated in the text.

Now

$$\frac{d}{d\gamma} \left[\frac{5}{4} q^3 p + \frac{3}{8} pq - \gamma \left(\frac{3}{8} + p^2 q^2 \right) \right] = -\frac{1}{2} [1 - 2 \cos 2\gamma + \cos 4\gamma + \gamma \sin 4\gamma],$$

and with the above value of γ is equal to $-0{\cdot}1355014$.

Also, with this value of γ the coefficient of k^2 is $0{\cdot}0160432$; so that

$$0{\cdot}0160432 \sin (\gamma - 54° \; 21' \; 27'') = 0{\cdot}1355014 k^2,$$

or,
$$\sin^2 \alpha = 10^{\cdot 9266528} \sin (\gamma - 54° \; 21' \; 27''),$$

which is the equation (21) of the text.

Again $\quad \int_0^\gamma \frac{q^2}{\Delta} d\gamma = -\frac{1}{2} qp + \frac{1}{2} \gamma + \frac{1}{2} k^2 \left(-\frac{1}{4} q^3 p - \frac{3}{8} qp + \frac{3}{8} \gamma \right),$

$$p^2 \int \frac{Q^2}{\Delta} d\gamma = qp - (1 - q^2) \gamma + \frac{1}{2} k^2 \left(q^3 p + \frac{3}{2} qp (1 - q^2) - \frac{3}{2} \gamma (1 - q^2) \right)$$

$$= qp - \gamma + q^2 \gamma + \frac{1}{2} k^2 \left(-\frac{1}{2} q^3 p + \frac{3}{2} qp - \frac{3}{2} \gamma + \frac{3}{2} q^2 \gamma \right).$$

Therefore

$$\int \frac{q^2}{\Delta} d\gamma - p^2 \int \frac{Q^2}{\Delta} d\gamma = -\frac{3}{2} qp + \left(\frac{3}{2} - q^2 \right) \gamma + \frac{1}{2} k^2 \left[\frac{1}{4} q^3 p - \frac{15}{8} qp + \left(\frac{15}{8} - \frac{3}{2} q^2 \right) \gamma \right].$$

Now $\qquad \Delta = 1 - \frac{1}{2} k^2 q^2 - \dots$

Approximate Solutions of the Problem—continued.

Therefore

$$\Delta\left[\int\frac{q^2}{\Delta}d\gamma-p^2\int\frac{Q^2}{\Delta}d\gamma\right]=-\frac{3}{2}qp+\left(\frac{3}{2}-q^2\right)\gamma$$

$$+\frac{1}{2}k^2\left[\frac{7}{4}q^3p-\frac{15}{8}qp+\left(\frac{15}{8}-3q^2+q^4\right)\gamma\right]$$

$$=p^2\left[-\frac{3}{2}Q+\left(\frac{3}{2}+\frac{1}{2}Q^2\right)\gamma\right]+\frac{1}{2}k^2p^2\left[\frac{7}{4}q^2Q-\frac{15}{8}Q+\left(\frac{15}{8}-\frac{9}{8}Q^2+q^2Q^2\right)\gamma\right]. \quad (j)$$

Hence from (*c*) and (*j*)

$$\frac{\omega^2}{4\pi}=\frac{1}{2Q^3}[(3+Q^2)\gamma-3Q]+\frac{1}{2}\frac{k^2}{Q^3}\left[\frac{7}{4}q^2Q-\frac{15}{8}Q+\left(\frac{15}{8}-\frac{9}{8}Q^2+q^2Q^2\right)\gamma\right],$$

which is the equation (22) of the text.

Secondly, let k be nearly unity and g small—

Then we require the following definite integrals :—

$$\int\frac{q^{n+1}}{p^n}d\gamma=\frac{1}{n-1}qQ^{n-1}-\frac{n}{n-1}\int\frac{q^{n-1}}{p^{n-2}}d\gamma. \quad (k)$$

$$\int\frac{q^2}{p}d\gamma=-q+\Lambda. \quad (l)$$

Now

$$\Delta^2=1-k^2q^2=p^2(1+g^2q^2p^{-2}).$$

$$\frac{1}{\Delta^3}=\frac{1}{p^3}\left(1-\frac{3}{2}g^2\frac{q^2}{p^2}+\dots\right).$$

$$\frac{q^4}{\Delta^3}=\frac{q^4}{p^3}-\frac{3}{2}g^2\frac{q^6}{p^5}+\dots$$

By (*k*) and (*l*)—

$$\int\frac{q^4}{\Delta^3}d\gamma=\frac{1}{2}qQ^2+\frac{3}{2}q-\frac{3}{2}\Lambda-\frac{3}{2}g^2\left[\frac{1}{4}qQ^4-\frac{5}{4\,.\,2}qQ^2-\frac{5\,.\,3}{4\,.\,2}q+\frac{5\,.\,3}{4\,.\,2}\Lambda\right]\dots$$

$$=\frac{3}{2}q\left[1+\frac{1}{3}Q^2+g^2\left(\frac{15}{8}+\frac{5}{8}Q^2-\frac{1}{4}Q^4\right)\right]-\frac{3}{2}\Lambda\left(1+\frac{15}{8}g^2\right)\dots$$

Now

$$\frac{\Delta^2}{p^2}=1+g^2Q^2,$$

therefore

$$\frac{\Delta^2}{p^2}\int\frac{q^4}{\Delta^3}d\gamma=\frac{3}{2}q\left[1+\frac{1}{3}Q^2+g^2\left(\frac{15}{8}+\frac{13}{8}Q^2+\frac{1}{12}Q^4\right)\right]-\frac{3}{2}\Lambda\left[1+g^2\left(\frac{15}{8}+Q^2\right)\right]\dots$$

But

$$\frac{1}{p^2}=1+Q^2,$$

therefore

$$\frac{\Delta^2}{p^4}\int\frac{q^4}{\Delta^3}d\gamma=\frac{3}{2}q\left[1+\frac{4}{3}Q^2+\frac{1}{3}Q^4+g^2\left(\frac{15}{8}+\frac{28}{8}Q^2+\frac{41}{24}Q^4+\frac{1}{12}Q^6\right)\right]$$

$$-\frac{3}{2}\Lambda\left[1+Q^2+g^2\left(\frac{15}{8}+\frac{23}{8}Q^2+Q^4\right)\right]. \quad (m)$$

Approximate Solutions of the Problem—continued.

Again, $$\frac{1}{\Delta}=\frac{1}{p}\left(1-\frac{1}{2}g^2\frac{q^2}{p^2}\dots\right).$$

$$\frac{Q^2}{\Delta}=\frac{q^4}{p^3}+\frac{q^2}{p}-\frac{1}{2}g^2\left(\frac{q^6}{p^5}+\frac{q^4}{p^3}\right)+\dots$$

$$\int\frac{Q^2}{\Delta}d\gamma=\left(\frac{1}{2}qQ^2+\frac{3}{2}q-\frac{3}{2}\Lambda\right)+(-q+\Lambda)-\frac{1}{2}g^2\left[\left(\frac{1}{4}qQ^4-\frac{5}{4.2}qQ^2-\frac{5.3}{4.2}q+\frac{5.3}{4.2}\Lambda\right)+\left(\frac{1}{2}qQ^2+\frac{3}{2}q-\frac{3}{2}\Lambda\right)\right]\dots$$

$$=\frac{1}{2}q(1+Q^2)+\frac{1}{2}qg^2\left(\frac{3}{8}+\frac{1}{8}Q^2-\frac{1}{4}Q^4\right)-\frac{1}{2}\Lambda\left(1+\frac{3}{8}g^2\right)\dots$$

$$Q^2\int\frac{Q^2}{\Delta}d\gamma=\frac{3}{2}q\left[\frac{1}{3}Q^2+\frac{1}{3}Q^4+g^2\left(\frac{1}{8}Q^2+\frac{1}{24}Q^4-\frac{1}{12}Q^6\right)\right]-\frac{3}{2}\Lambda\left(\frac{1}{3}Q^2+\frac{1}{8}g^2Q^2\right). \quad (n)$$

The equation (b) for determining the axes is

$$\frac{\Delta^2}{p^4}\int\frac{q^4}{\Delta^3}d\gamma=Q^2\int\frac{Q^2}{\Delta}d\gamma.$$

Hence equating (m) and (n) and dividing by $\frac{3}{2}q$ we have

$$1+Q^2+g^2\left(\frac{15}{8}+\frac{27}{8}Q^2+\frac{5}{3}Q^4+\frac{1}{6}Q^6\right)-\frac{\Lambda}{q}\left[1+\frac{2}{3}Q^2+g^2\left(\frac{15}{8}+\frac{11}{4}Q^2+Q^4\right)\right]\dots=0,$$

which is the equation (23) in the text.

The equation (c) for ω may be written

$$\frac{\omega^2}{4\pi}=\frac{\Delta}{Q^3}\left[(1+Q^2)\int\frac{q^2}{\Delta}d\gamma-\int\frac{Q^2}{\Delta}d\gamma\right].$$

Now $$\int\frac{q^2}{\Delta}d\gamma=\int\left[\frac{q^2}{p}-\frac{1}{2}g^2\left(\frac{q^2}{p}+\frac{q^4}{p^3}\right)\right]d\gamma$$

$$=\left(1-\frac{1}{2}g^2\right)(q-\Lambda)-\frac{1}{2}g^2\left[\frac{1}{2}qQ^2+\frac{3}{2}q-\frac{3}{2}\Lambda\right]$$

$$=q\left\{\frac{\Lambda}{q}-1+\frac{1}{2}g^2\left[\frac{1}{2}\Lambda-\frac{1}{2}-\frac{1}{2}Q^2\right]\right\}.$$

$$(1+Q^2)\int\frac{q^2}{\Delta}d\gamma=q\left\{\left(\frac{\Lambda}{q}-1\right)(1+Q^2)+\frac{1}{2}g^2\left[\frac{1}{2}\Lambda(1+Q^2)-\frac{1}{2}(1+Q^2)^2\right]\right\}$$

$$\int\frac{Q^2}{\Delta}d\gamma=q\left\{\frac{1}{2}(1+Q^2)-\frac{1}{2}\Lambda+\frac{1}{2}g^2\left[\frac{3}{8}+\frac{1}{8}Q^2-\frac{1}{4}Q^4-\frac{3}{8}\Lambda\right]\right\}$$

$$(1+Q^2)\int\frac{q^2}{\Delta}d\gamma-\int\frac{Q^2}{\Delta}d\gamma=q\left\{\frac{\Lambda}{q}\left(\frac{3}{2}+Q^2\right)-\frac{3}{2}(1+Q^2)+\frac{1}{2}g^2\left[\frac{\Lambda}{q}\left(\frac{7}{8}+\frac{1}{2}Q^2\right)-\frac{7}{8}-\frac{9}{8}Q^2-\frac{1}{4}Q^4\right]\right\}.$$

If α be infinitely small, so that the Jacobian ellipsoid of three unequal axes becomes in the limit an ellipsoid of revolution, we have γ given by

$$\gamma = \frac{\sin 2\gamma - \frac{5}{16}\sin 4\gamma}{1 - \frac{1}{4}\cos 4\gamma}.$$

The solution of this is $\gamma = 54° \ 21' \ 27''$.

If we write $\tan\gamma = f$, as in Thomson and Tait's 'Natural Philosophy,' §778′, this equation becomes

$$\frac{\tan^{-1} f}{f} = \frac{1 + \frac{13}{3} f^2}{1 + \frac{14}{3} f^2 + f^4},$$

which is the equation (9), §778′, of that work.

The ellipsoid of revolution of which the eccentricity is sin 54° 21′ 27″ belongs to the revolutional series of figures of equilibrium, and is the starting point of the Jacobian series of figures. As shown by Sir William Thomson, it is the flattest revolutional figure which is dynamically stable. The Jacobian figures of equilibrium are initially stable, and as stated by M. Poincaré,* there is for this value of γ a crossing point of the two series, and an exchange of stabilities.

If α be small, it appears that $\sin\alpha$ is given by

$$\sin^2\alpha = 10^{\cdot 9266528} \sin(\gamma - 54° \ 21' \ 27''), \quad . \quad . \quad . \quad (21)$$

and ω by

$$\frac{\omega^2}{4\pi} = \tfrac{1}{2}\cot^3\gamma\left[(3 + \tan^2\gamma)\gamma - 3\tan\gamma\right]$$
$$+ \tfrac{1}{2}\frac{\sin^2\alpha}{\tan^3\gamma}\left[(\tfrac{7}{4}\sin^2\gamma - \tfrac{15}{8})\tan\gamma + \gamma(\tfrac{15}{8} - (\tfrac{9}{8} - \sin^2\gamma)\tan^2\gamma)\right] \ . \quad (22)$$

Approximate Solutions of the Problem—continued.

Now
$$\Delta = p\left(1 + \frac{1}{2}g^2 Q^2 \ldots\right).$$

Therefore

$$\Delta\left[(1+Q^2)\int\frac{q^2}{\Delta}d\gamma - \int\frac{Q^2}{\Delta}d\gamma\right] = pq\left\{\frac{\Lambda}{q}\left(\frac{3}{2} + Q^2\right) - \frac{3}{2}(1+Q^2)\right.$$
$$\left. + \frac{1}{2}g^2\left[\Lambda\left(\frac{7}{8} + 2Q^2 + Q^4\right) - \frac{7}{8} - \frac{21}{8}Q^2 - \frac{7}{4}Q^4\right]\right\}.$$

Hence

$$\frac{\omega^2}{4\pi} = \frac{pq}{Q^3}\left\{\frac{\Lambda}{q}\left(\frac{3}{2} + Q^2\right) - \frac{3}{2}(1+Q^2) + \frac{1}{2}g^2\left[\frac{\Lambda}{q}\left(\frac{7}{8} + 2Q^2 + Q^4\right) - \frac{7}{8} - \frac{21}{8}Q^2 - \frac{7}{4}Q^4\right]\right\},$$

which is the equation (24) in the text.

* "Sur l'Équilibre d'une Masse de Fluide, &c.," 'Acta Mathematica,' 7, 1886.

These formulæ are, it must be admitted, of but little use, since it would be necessary to take in higher powers of $\sin^2\alpha$ to obtain results for a variation of γ of more than 1°.

If α be near 90°, so that $\cos\alpha$ is small, the approximate equation between α and γ is

$$1+\tan^2\gamma+\cos^2\alpha\left(\tfrac{15}{8}+\tfrac{27}{8}\tan^2\gamma+\tfrac{5}{3}\tan^4\gamma+\tfrac{1}{6}\tan^6\gamma\right)$$

$$-\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma\right)\cdot\left[1+\tfrac{2}{3}\tan^2\gamma+\cos^2\alpha\left(\tfrac{15}{8}+\tfrac{11}{4}\tan^2\gamma+\tan^4\gamma\right)\right]$$
$$=0. \quad . \quad . \quad . \quad (23)$$

And ω is given by

$$\frac{\omega^2}{4\pi}=\tfrac{1}{2}\frac{\sin 2\gamma}{\tan^3\gamma}\left\{\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma\right)\cdot\left(\tfrac{3}{2}+\tan^2\gamma\right)-\tfrac{3}{2}\sec^2\gamma\right.$$

$$+\tfrac{1}{2}\cos^2\alpha\left[\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma\right)\cdot\left(\tfrac{7}{8}+2\tan^2\gamma+\tan^4\gamma\right)-\tfrac{7}{8}-\tfrac{21}{8}\tan^2\gamma\right.$$

$$\left.\left.-\tfrac{7}{4}\tan^4\gamma\right]\right\} \quad . \quad . \quad (24)$$

We shall return later to a modification of (24) which will be applicable to very long ellipsoids of equilibrium.

Besides the angular velocity and the axes of the ellipsoid, the other important functions are the moment of momentum, the kinetic energy of rotation, and the intrinsic energy of the mass. In order to express these numerically we must adopt a unit of length, and it will be convenient to take a, where

$$\mathrm{a}^3=abc=a^3\cos\beta\cos\gamma.$$

Thus $$a=\mathrm{a}(\sec\beta\sec\gamma)^{\frac{1}{3}}.$$

Let σ be the density of the fluid which has hitherto been treated as unity, and let $(\tfrac{4}{3}\pi\sigma)^{\frac{3}{2}}\mathrm{a}^5\mu$, $(\tfrac{4}{3}\pi\sigma)^2\mathrm{a}^5\epsilon$ be the moment of momentum and kinetic energy, then

$$(\tfrac{4}{3}\pi\sigma)^{\frac{3}{2}}\mathrm{a}^5\mu=\tfrac{1}{5}m(a^2+b^2)\omega=\tfrac{4}{15}\pi\sigma\mathrm{a}^5(\sec\beta\sec\gamma)^{\frac{2}{3}}(1+\cos^2\beta)(4\pi\sigma)^{\frac{1}{2}}\left(\frac{\omega^2}{4\pi\sigma}\right)^{\frac{1}{2}}.$$

Thus $$\mu=\tfrac{1}{5}\sqrt{3}(\sec\beta\sec\gamma)^{\frac{2}{3}}(1+\cos^2\beta)\left(\frac{\omega^2}{4\sigma}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (25)$$

The function (25) is the quantity which will be tabulated.

Again $$(\tfrac{4}{3}\pi\sigma)^2\mathrm{a}^5\epsilon=\tfrac{1}{2}(\tfrac{4}{3}\pi\sigma)^{\frac{3}{2}}\mathrm{a}^5\mu\,.\,\omega=\tfrac{1}{2}\sqrt{3}(\tfrac{4}{3}\pi\sigma)^2\mathrm{a}^5\,.\,\mu\left(\frac{\omega^2}{4\pi\sigma}\right)^{\frac{1}{2}},$$

so that $$\epsilon=\tfrac{1}{2}\sqrt{3}\mu\left(\frac{\omega^2}{4\pi\sigma}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (26)$$

The function (26) is the quantity which will be tabulated.

Thus in the tables the unit of moment of momentum is taken as $(\frac{4}{3}\pi\sigma)^{\frac{3}{2}}\mathrm{a}^5$, or $m^{\frac{3}{2}}\mathrm{a}^{\frac{1}{2}}$, and the unit of energy as $(\frac{4}{3}\pi\sigma)^2\mathrm{a}^5$ or m^2/a.

It remains to evaluate the intrinsic energy, or the energy required to expand the ellipsoid against its own gravitation, into a condition of infinite dispersion.

If dt be an element of volume, then this energy is

$$-\tfrac{1}{2}\iiint \mathrm{V}\sigma dt$$

integrated throughout the ellipsoid.

This will be denoted by $(\frac{4}{3}\pi\sigma)^2\mathrm{a}^5(i-1)$, or $m^2\mathrm{a}^{-1}(i-1)$, so that i will be positive.

Now $\mathrm{V}=Lx^2+My^2+Nz^2+P$, and if we denote by A, B, C, the principal moments of inertia of the ellipsoid, we have

$$\iiint x^2\sigma\, dt=\tfrac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})=\tfrac{1}{5}ma^2,$$

and similarly, $\iiint y^2\sigma\, dt=\frac{1}{5}mb^2$, $\iiint z^2\sigma\, dt=\frac{1}{5}mc^2$.

Also $$\iiint \sigma dt=m.$$

Hence $$\frac{m^2}{\mathrm{a}}(i-1)=\tfrac{1}{10}m\,[La^2+Mb^2+Nc^2+5P]$$

$$=\tfrac{1}{10}m\mathrm{a}^2(\sec\beta\sec\gamma)^{\frac{2}{3}}[L+M\cos^2\beta+N\cos^2\gamma+5Pa^{-2}].$$

But if we take the values of L, M, N given in (15), and note that

$$P=\pi a^3\cos\beta\cos\gamma\,.\,\frac{2}{a\sin\gamma}\mathrm{F},$$

it easily follows that

$$L+M\cos^2\beta+N\cos^2\gamma+Pa^{-2}=0.$$

Hence $$\frac{m^2}{\mathrm{a}}(i-1)=-\tfrac{2}{5}m\mathrm{a}^2(\sec\beta\sec\gamma)^{\frac{2}{3}}\,.\,Pa^{-2}$$

$$=-\tfrac{2}{5}m\mathrm{a}^2(\sec\beta\sec\gamma)^{\frac{2}{3}}\,.\,\tfrac{3}{4}\frac{m}{a^2}\cdot\frac{2}{a\sin\gamma}\mathrm{F}$$

$$=-\tfrac{3}{5}\frac{m^2}{\mathrm{a}}\frac{(\cos\beta\cos\gamma)^{\frac{1}{3}}}{\sin\gamma}\mathrm{F}.$$

Therefore $$i=1-\tfrac{3}{5}\frac{(\cos\beta\cos\gamma)^{\frac{1}{3}}}{\sin\gamma}\mathrm{F}. \qquad (27)$$

For a sphere γ becomes infinitely small, and F becomes equal to γ, so that $\mathrm{F}/\sin\gamma=1$. Thus $i-1=-\frac{3}{5}$. Therefore the exhaustion of energy of a sphere of radius a is $\frac{3}{5}m^2/\text{a}$; which is the known result. For an ellipsoid of revolution $\alpha=0$, and $\beta=0$, and $\text{F}=\gamma$; so that

$$i=1-\tfrac{3}{5}\gamma\frac{\cos^{\frac{1}{3}}\gamma}{\sin\gamma}.$$

The function (27) is the quantity tabulated below. It seemed preferable to tabulate a positive quantity, and it is on this account that the intrinsic energy corresponding to the infinitely long ellipsoid is entered as unity.

Having now obtained all the necessary formulæ, we may proceed to consider the solution of the problem.

We have to solve

$$\sec^2\alpha\sec^2\zeta\,\text{E}-(2\text{F}-\text{E})-\tan\zeta\sec^2\alpha\sec^2\delta=0, \quad . \quad . \quad (28)$$

where $\tan\zeta=\sin\alpha\tan\beta\cos\gamma$, $\tan\delta=\sin\beta=\sin\alpha\sin\gamma$,

and $$\text{F}=\int_0^\gamma\frac{d}{\cos\beta}, \qquad \text{E}=\int_0^\gamma\cos\beta\,d\gamma.$$

The axes of the ellipsoid are

$$\frac{a}{\text{a}}=(\sec\beta\sec\gamma)^{\frac{1}{3}}, \qquad \frac{b}{\text{a}}=\frac{a}{\text{a}}\cos\beta, \qquad \frac{c}{\text{a}}=\frac{a}{\text{a}}\cos\gamma\,. \quad . \quad (29)$$

If e_1, e_2, e_3 are the eccentricities of the sections through *ca*, *cb*, *ab* respectively, we have

$$e_1=\sin\beta, \qquad e_2=\sin\gamma, \qquad e_3=\cos\alpha\sin\gamma\sec\beta. \quad . \quad (30)$$

Having obtained the solution, we have to compute

$$\frac{w^2}{4\pi\sigma}=\cot\beta\,\text{cosec}\,\beta\cot\gamma(\text{F}-\text{E})+\cot^3\beta\cos\beta\sec^2\alpha\text{E}-\cos^2\beta\cot^2\gamma\sec^2\alpha^* \quad . \quad . \quad . \quad (31)$$

Then we next compute μ and ϵ and i from the formulæ (25), (26), (27).

The functions F and E are tabulated in Table IX of the second volume of Legendre's 'Traité des Fonctions Elliptiques,' in a table of double entry for α and γ for each degree.

The solution of (28) by trial and error was laborious, as it was necessary to work with all the accuracy attainable with logarithms of seven figures.

The method adopted was to choose an arbitrary value of γ, and

* As stated above, some of the computations were actually made from the formula (20).

Solutions of Jacobi's Problems.

Auxiliary angles.			Axes.			Eccentricities of sections.			Ang. vel.	Mom. of momentum.	Energy.		
γ.	α.	β.	a/a.	b/a.	c/a.	(ac.) e_1.	(bc.) e_2.	(ab.) e_3.	$\omega^2/4\pi\sigma$.	μ.	Kinetic. ϵ.	Intrinsic. i.	Total. E.
54° 21′ 27″	0° 0′	0° 0′	1·1972	1·1972	·6977	·81267	·81267	·0000	·09356	·30375	·08046	·41495	·49541
55	$17\frac{3}{4}$	$14\frac{1}{4}$	1·216	1·179	·698	·819	·806	·246	·094	·304	—	—	—
57	$34\frac{3}{4}$	$28\frac{1}{2}$	1·279	1·123	·696	·839	·784	·478	·093	·306	—	—	—
60	49 7	40 54	1·3831	1·0454	·6916	·8660	·7500	·6547	·09060	·3134	·0817	·4188	·5005
65	64 19	54 46	1·6007	·9235	·6765	·9063	·6807	·8168	·08295	·3407	·0850	·4394	·5244
70	74 12	64 43	1·899	·8111	·6494	·9397	·5991	·9042	·07047	·3920	·0901	·4189	·5390
75	81 4	72 36	2·346	·7019	·6072	·9659	·5014	·9542	·0536	·4809	·0964	·4808	·5772
80	86	$79\frac{1}{4}$	3·136	·5858	·5445	·9848	·3684	·9824	·0334	·644	·1019	·5334	·6353
85	$88\frac{5}{6}$	$84\frac{5}{6}$	5·04	·45	·44	·996	·228	·996	·013	1·016	·101	·645	·746
90	90	90	∞	·00	·00	1·000	·000	1·000	·000	∞	·000	1·000	1·000

N.B.—*The moment of momentum of the system is* $(\frac{4}{3}\pi\sigma)^{\frac{3}{2}}a^5.\mu$, *or* $m^{\frac{3}{2}}a^{\frac{1}{2}}\mu$; *the kinetic energy is* $(\frac{4}{3}\pi\sigma)^2a^5.\epsilon$, *or* $m^2a^{-1}\epsilon$, *and the intrinsic energy is* $(\frac{4}{3}\pi\sigma)^2a^5.(i-1)$, *or* $m^2a^{-1}(i-1)$, *but in the above table unity has been added to make the results positive;* $E=\epsilon+i$.

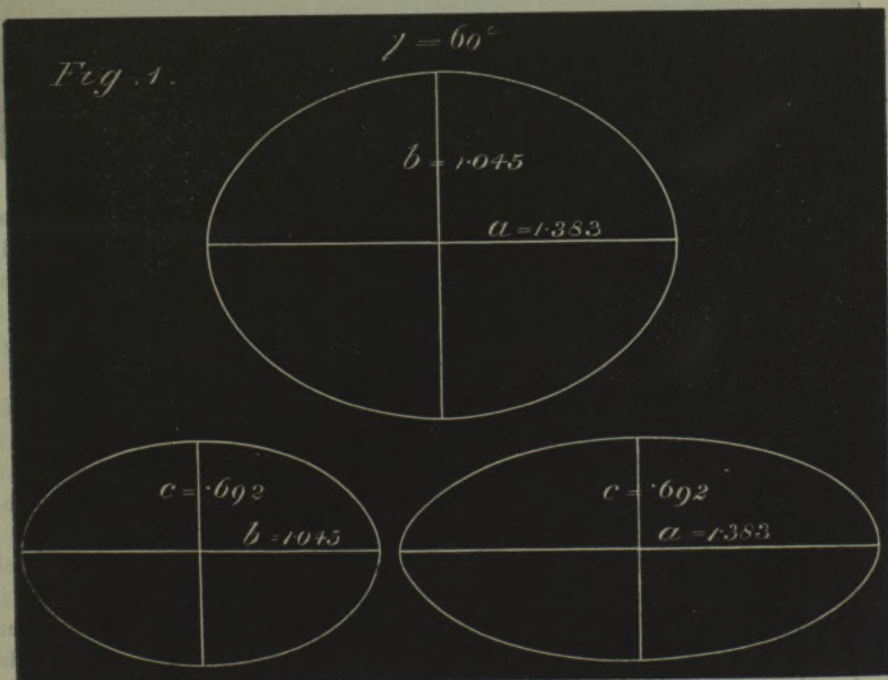
then by trial and error to find two values of α one degree apart, one of which made the left-hand side of (28) positive, and the other negative.

The smallest value of γ is 54° 21′ 27″, but after that value integral degrees for γ were always chosen.

The solutions for $\gamma=55°$ and 57° could not be found very exactly from the elliptic integrals with logarithms of only seven figures, but the solutions were confirmed by the approximate formulæ (21) and (22). The solution for $\gamma=80°$ was confirmed by the approximate formulæ (23) and (24), and that for $\gamma=85°$ was only computed therefrom, since when $\gamma=80°$ the approximate formula gave nearly identical results with the exact one.

The solution obtained is embodied in the table opposite. The first three columns give the auxiliary angles γ, α, β, from which the remaining results are computed.

As a graphical result is much more intelligible than a numerical one, I have given two figures, showing the three principal sections in two cases, namely, where $\gamma=60°$, and $\gamma=75°$. For these figures a is taken as 2 cm., so that the volume of fluid is $\frac{4}{3}\pi \times 2^3$ cubic cm.



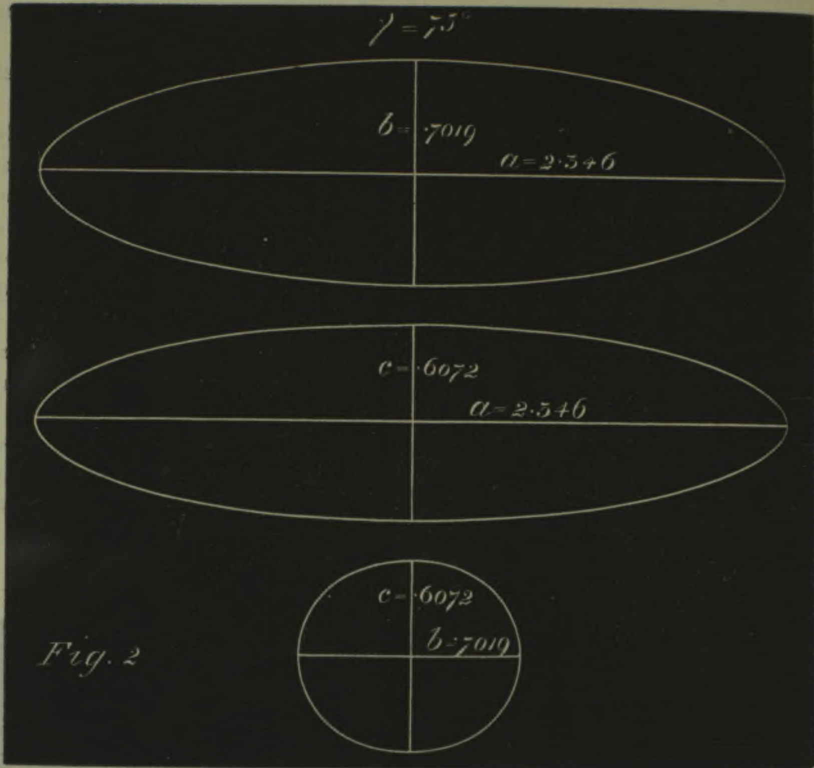


Fig. 2

It will be noticed that the longer the ellipsoid the slower it rotates. It is interesting to observe that while the angular velocity continually diminishes, the moment of momentum continually increases. The long ellipsoids are very nearly ellipsoids of revolution about an axis perpendicular to that of rotation. Thus in fig. 2 the section through b and c is not much flattened.

The most remarkable point is that there is a maximum of kinetic energy when a/a is about 3, or when the length of the ellipsoid is about five times its diameter. However, notwithstanding this maximum of kinetic energy, the total energy always increases with the length of the ellipsoid.

The kinetic energy is the product of two factors, one of which always increases, and the other of which always diminishes; thus it is obvious that it must have a maximum. The result was, however, quite unforeseen, and it seems worth while to obtain simpler formulæ for the case of the long ellipsoids. This may be done by taking as the parameter a/a, or the length of the ellipsoid, instead of γ.

From the table we see that in the later entries β is very nearly equal to γ, and that α becomes very nearly equal to 90°. Hence we may put $\alpha=90°$, and $\beta=\gamma$.

Thus, approximately,

$$\frac{a}{\mathrm{a}} = (\sec\beta\sec\gamma)^{\frac{1}{3}} = (\sec\gamma)^{\frac{2}{3}}$$

and

$$\cos\gamma = \left(\frac{\mathrm{a}}{a}\right)^{\frac{3}{2}}, \qquad \gamma = \tfrac{1}{2}\pi - \left(\frac{\mathrm{a}}{a}\right)^{\frac{3}{2}}.$$

The axes of the ellipsoid are

$$a,\ \left(\frac{\mathrm{a}^3}{a}\right)^{\frac{1}{2}},\ \left(\frac{\mathrm{a}^3}{a}\right)^{\frac{1}{2}}.$$

Now if in formula (24) we only retain the higher powers of $\tan\gamma$, we have

$$\frac{\omega^2}{4\pi\sigma} = \frac{\sin\gamma\cos\gamma}{\tan\gamma}\left[\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\right) - \tfrac{3}{2}\right]$$

$$= \tfrac{3}{2}\,\frac{\cos^2\gamma}{\sin\gamma}\left[\tfrac{2}{3}\log_e\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\right) - \sin\gamma\right].$$

But

$$\log_e\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\right) = \log_e\frac{1+\sin\gamma}{\cos\gamma} = \log_e 2 + \tfrac{3}{2}\log_e\frac{a}{\mathrm{a}}.$$

Therefore writing $1-\frac{2}{3}\log_e 2 = \mathrm{C}$, so that $\mathrm{C}=0{\cdot}3573$, we have

$$\frac{\omega^2}{4\pi\sigma} = \tfrac{3}{2}\left(\frac{\mathrm{a}}{a}\right)^3\left[\log_e\frac{a}{\mathrm{a}} - \mathrm{C}\right].$$

If we put $a/\mathrm{a}=5{\cdot}042$, this formula gives $\omega^2/4\pi\sigma = 0{\cdot}01264$. The full value in the preceding tables was 0·0131; thus even with so short an ellipsoid as this, the results agree within 4 per cent. With rougher approximation we have

$$\frac{\omega^2}{4\pi\sigma} = \tfrac{3}{2}\left(\frac{\mathrm{a}}{a}\right)^3\log_e\frac{a}{\mathrm{a}},$$

of which the limit, when a is large, is zero.

For the moment of momentum we have

$$\mu = \frac{3^{\frac{1}{2}}}{5}(\sec\beta\sec\gamma)^{\frac{2}{3}}(1+\cos^2\beta)\left(\frac{\omega^2}{4\pi\sigma}\right)^{\frac{1}{2}}$$

$$= \frac{3}{5.2^{\frac{1}{2}}}\left(\frac{a}{\mathrm{a}}\right)^{\frac{1}{2}}\left(1+\frac{\mathrm{a}^3}{a^3}\right)\left(\log_e\frac{a}{\mathrm{a}} - \mathrm{C}\right)^{\frac{1}{2}},$$

or, with rougher approximation,

$$\mu = \frac{3}{5.2^{\frac{1}{3}}}\left(\frac{a}{\mathrm{a}}\right)^{\frac{1}{3}}\left(\log_e\frac{a}{\mathrm{a}}\right)^{\frac{1}{2}},$$

of which the limit is infinite.

Again,
$$\epsilon = \frac{3^{\frac{1}{2}}}{2}\mu\left(\frac{\omega^2}{4\pi\sigma}\right)$$

$$= \tfrac{9}{20}\frac{\mathrm{a}}{a}\left[1+\frac{\mathrm{a}^3}{a^3}\right]\left[\log_e\frac{a}{\mathrm{a}}-\mathrm{C}\right].$$

Now the function $\frac{\mathrm{a}}{a}\left(\log_e\frac{a}{\mathrm{a}}-\mathrm{C}\right)$ has a maximum, when $\log_e\frac{a}{\mathrm{a}}=1+\mathrm{C}$ $=1{\cdot}3573$, that is when $\frac{\mathrm{a}}{a}=1{\cdot}696$.

On comparison with our tables it is obvious that the approximation is bad, and that the true solution for a maximum is considerably different from the above. Nevertheless this investigation shows that there is actually a maximum of kinetic energy.

Since
$$\mathrm{F}=\log_e\cot\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma\right)=\tfrac{3}{2}\left[\log_e\frac{a}{\mathrm{a}}+\tfrac{2}{3}\log_e 2\right],$$

we have

$$i=1-\tfrac{3}{5}\frac{(\cos\beta\cos\gamma)^{\frac{1}{3}}}{\sin\gamma}\mathrm{F}=1-\tfrac{9}{10}\left(\frac{\mathrm{a}}{a}\right)^3\left[\log_e\frac{a}{\mathrm{a}}+\tfrac{2}{3}\log_e 2\right].$$

If we like we may express these several results in terms of the minor and major axes of the ellipsoid, for $b=c=\frac{\mathrm{a}^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, and therefore $\mathrm{a}^3=c^2a$.

Thus
$$\left.\begin{aligned}
\frac{\omega^2}{4\pi\sigma} &= \frac{c^2}{a^2}\left[\log_e\frac{a}{c}-\tfrac{3}{2}\mathrm{C}\right].\\
\mu &= \frac{3^{\frac{1}{2}}}{5}\left(\frac{a}{c}\right)^{\frac{1}{3}}\left[1+\frac{c^2}{a^2}\right]\left[\log_e\frac{a}{c}-\tfrac{3}{2}\mathrm{C}\right]^{\frac{1}{2}}.\\
\epsilon &= \tfrac{3}{10}\left(\frac{c}{a}\right)^{\frac{4}{3}}\left(1+\frac{c^2}{a^2}\right)\left(\log_e\frac{a}{c}-\tfrac{3}{2}\mathrm{C}\right).\\
i &= 1-\tfrac{3}{5}\,\frac{a^2}{c^2}\log_e\frac{2a}{c}.
\end{aligned}\right\}$$