Fig. 25. Plasmodium (*Cancer Pagurus*).
Fig. 26. Corpuscles of *Asteracanthion vulgare*, freshly drawn.
Figs. 27—28. Union of a group of corpuscles of *Asteracanthion vulgare*.
Fig. 29. Portion of a plasmodium produced by the union of the finely granular corpuscles of *Echinus sphæra*, showing distinct endoplasm containing the coarsely granular and the coloured corpuscles, and ectoplasm sending out filamentous pseudopodia, which unite with those of free corpuscles.
Figs. 30—33. *Phonergates vorax*, from "Zeitsch. f. Wiss. Zool." Bd. XXX. 1878. Taf. II, figs. 54—57.

All the figures drawn with Verick, Oc. 2, Obj. 7.

IV. "On the Analytical Expressions which give the History of a Fluid Planet of Small Viscosity, attended by a Single Satellite." By G. H. DARWIN, F.R.S. Received March 6, 1880.

In a series of papers read from time to time during the past two years before the Royal Society, I have investigated the theory of the tides raised in a rotating viscous spheroid, or planet, by an attendant satellite, and have also considered the secular changes in the rotation of the planet, and in the revolution of the satellite. Those investigations were intended to be especially applicable to the case of the earth and moon, but the friction of the solar tides was found to be a factor of importance, so that in a large part of those papers it became necessary to conceive the planet as attended by two satellites.

The differential equations which gave the secular changes in the system were rendered very complex by the introduction of solar disturbance, and I was unable to integrate them analytically; the equations were accordingly treated by a method of numerical quadratures, in which all the data were taken from the earth, moon, and sun. This numerical treatment did not permit an insight into all the various effects which might result from frictional tides, and an analytical solution, applicable to any planet and satellite, is desirable.

In the present paper such an analytical solution is found, and is interpreted graphically. But the problem is considered from a point of view which is at once more special and more general than that of the previous papers.

The point of view is more general in that the planet may here be conceived to have any density and mass whatever, and to be rotating with any angular velocity, provided that the ellipticity of figure is not large, and that the satellite may have any mass, and may be revolving about its planet, either consentaneously with or adversely to the planetary rotation. On the other hand, the problem here considered is more special in that the planet is supposed to be a spheroid of fluid of small viscosity; that the obliquity of the planet's equator, the inclination and the eccentricity of satellite's orbit to the plane of reference are treated as being small, and, lastly, it is supposed that the planet is only attended by a single satellite.

The satellite itself is treated as an attractive particle, and the planet is supposed to be homogeneous.

The notation adopted is made to agree as far as possible with that of a previous paper, in which the subject was treated from a similarly general point of view, but where it was supposed that the equator and orbit were co-planar, and the orbit necessarily circular.*

The motion of the system is referred to the invariable plane, that is, to the plane of maximum moment of momentum.

The following is the notation adopted :—

For the planet :—

M = mass; a = mean radius; g = mean pure gravity; C = moment of inertia (neglecting ellipticity of figure); n = angular velocity of rotation; i = obliquity of equator to invariable plane, considered as small; $\mathfrak{g} = \frac{2}{5}g/a$.

For the satellite :—

m = mass; c = mean distance; Ω = mean motion; e = eccentricity of orbit, considered as small; j = inclination of orbit, considered as small; $\tau = \frac{3}{2}m/c^3$, where m is measured in the astronomical unit.

For both together :—

$\nu = M/m$, the ratio of the masses; $s = \frac{2}{5}[(a\nu/g)^2(1+\nu)]^{\frac{1}{3}}$; h = the resultant moment of momentum of the whole system; E = the whole energy, both kinetic and potential, of the system.

By a proper choice of the units of length, mass, and time, the notation may be considerably simplified.

Let the unit of length be such that $M+m$, when measured in the astronomical unit, may be equal to unity.

Let the unit of time be such that s or $\frac{2}{5}[a\nu/g)^2(1+\nu)]^{\frac{1}{3}}$ may be unity.

Let the unit of mass be such that C, the planet's moment of inertia, may be unity.

Then we have

$$\Omega^2c^3 = M+m = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now, if we put for g its value M/a^2, and for ν its value M/m, we have

$$s=\frac{2}{5}\left\{\left[\frac{aM}{m}\cdot\frac{a^2}{M}\right]^2\frac{M+m}{m}\right\}^{\frac{1}{3}}=\frac{2}{5}\frac{a^2}{m}, \text{ since } M+m \text{ is unity,}$$

and since s is unity, $m = \frac{2}{5}a^2$, when m is estimated in the astronomical unit.

* "Determination of the Secular Effects of Tidal Friction by a Graphical Method," "Proc. Roy. Soc.," No. 197, 1879.

Again, since $C = \frac{2}{5}Ma^2$, and since C is unity, therefore $M = \frac{5}{2}/a^2$, where M is estimated in the mass unit.

Therefore $Mm/(M+m)$ is unity, when M and m are estimated in the mass unit, with the proposed units of length, time, and mass.

According to the theory of elliptic motion, the moment of momentum of the orbital motion of the planet and satellite about their common centre of inertia is $\frac{Mm}{M+m}\Omega c^2\sqrt{1-e^2}$. Now it has been shown that the factor involving M is unity, and by (1) $\Omega c^2 = \Omega^{-\frac{1}{3}} = c^{\frac{1}{2}}$.

Hence, if we neglect the square of the eccentricity e, the m. of m. of orbital motion is numerically equal to $\Omega^{-\frac{1}{3}}$ or $c^{\frac{1}{2}}$.

Let $x = \Omega^{-\frac{1}{3}} = c^{\frac{1}{2}}$.

In this paper x, the moment of momentum of orbital motion, will be taken as the independent variable. In interpreting the figures given below it will be useful to remember that it is also equal to the square root of the mean distance.

The moment of momentum of the planet's rotation is equal to Cn; and since C is unity, n will be either the m. of m. of the planet's rotation, or the angular velocity of rotation itself.

With the proposed units $\tau = \frac{3}{2}m/c^3 = \frac{3}{5}a^2x^{-6}$, since $m = \frac{2}{5}a^2$; and $\mathfrak{g} = \frac{2}{5}g/a = \frac{2}{5}M/a^3 = \frac{2}{5}m.\ M/ma^3 = \frac{4}{25}\nu/a$.

Also $\tau^2/\mathfrak{g}$ (a quantity which occurs below) is equal to $\frac{9}{4}a^5/\nu x^{12}$.

Now let t be the time, and let $2f$ be the phase-retardation of the tide which I have elsewhere called the sidereal semi-diurnal tide of speed $2n$, which tide is known in the British Association Report on Tides as the faster of the two K tides.

Then if the planet be a fluid of small viscosity, the following are the differential equations which give the secular changes in the elements of the system:

$$\frac{dn}{dt} = -\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f\left(1-\frac{\Omega}{n}\right) \quad . \quad . \quad . \quad (2).$$

$$\frac{dx}{dt} = \frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f\left(1-\frac{\Omega}{n}\right) \quad . \quad . \quad . \quad (3).$$

$$\frac{di}{dt} = \frac{1}{4}\frac{\tau^2}{\mathfrak{g}}\sin 4f\left(\frac{i+j}{n}\right)\left(1-\frac{2\Omega}{n}\right) \quad . \quad . \quad . \quad (4).$$

$$\frac{dj}{dt} = -\frac{1}{4}\frac{\tau^2}{\mathfrak{g}}\sin 4f\,\frac{(i+j)}{x} \quad . \quad . \quad . \quad (5).$$

$$\frac{1}{e}\frac{de}{dt} = \frac{1}{4}\frac{\tau^2}{\mathfrak{g}}\sin 4f . \frac{1}{x}\left(11-\frac{18\Omega}{n}\right) \quad . \quad . \quad . \quad (6).$$

The first three of these equations are in effect established in my paper on the "Precession of a Viscous Spheroid,"* § 17, p. 497, eq. (80).

* "Phil. Trans.," Part II, 1879.

The suffix m^2 to the symbols i and N there indicates that the equations (80) only refer to the action of the moon, and as here we only have a single satellite, they are the complete equations. N is equal to n/n_0, so that n_0 disappears from the first and second of (80); also $\mu=1/sn_0\Omega_0^{\frac{1}{3}}$, and thus n_0 disappears from the third equation. $P=\cos i$, $Q=\sin i$, and, since we are treating i the obliquity as small, $P=1$, $Q=i$; also $\lambda=\Omega/n$; the ϵ of that paper is identical with the f of the present one; lastly ξ is equal to $\Omega_0^{\frac{1}{3}}\Omega^{-\frac{1}{3}}$, and since with our present units $s=1$, therefore $\mu\, d\xi/dt=d\Omega^{-\frac{1}{3}}/n_0dt=dx/n_0dt$.

With regard to the transformation of the first of (80) into (4) of the present paper, I remark that treating i as small $\frac{1}{4}PQ-\frac{1}{2}\lambda Q=\frac{1}{4}i(1-2\Omega/n)$, and introducing this transformation into the first of (80), equation (4) is obtained, except that i occurs in place of $(i+j)$. Now in the paper on the "Precession of a Viscous Spheroid" the inclination of the orbit of the satellite to the plane of reference was treated as zero, and hence j was zero; but I have proved in a paper "On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a tidally distorted Planet" (read before the Royal Society on December 18th, 1879, but as yet unpublished) that when we take into account the inclination of the orbit of the satellite, the P and Q on the right-hand sides of eq. (80) of "Precession" must be taken as the cosine and sine of $i+j$ instead of i. Equations (5) and (6) are proved in § 10, Part II, and § 25, Part V of the unpublished paper, and the reader is requested to take them as established.

The integrals of this system of equations will give the secular changes in the motion of the system under the influence of the frictional tides. The object of the present paper is to find an analytical expression for the solution, and to interpret that solution geometrically.

From equations (2) and (4) we have

$$i\frac{dn}{dt}+n\frac{di}{dt}=\frac{1}{4}\frac{\tau^2}{\mathfrak{g}}\sin 4f\left[-(i+j)+2j\left(1-\frac{\Omega}{n}\right)\right].$$

But from (3) and (5) $x\,dj/dt+j\,dx/dt$ is equal to the same expression; hence

$$i\frac{dn}{dt}+n\frac{di}{dt}=x\frac{dj}{dt}+j\frac{dx}{dt}.$$

The integral of this equation is $in=jx$.

or
$$\frac{i}{j}=\frac{x}{n} \quad \dots\dots\dots\dots \quad (7).$$

Equation (7) may also be obtained by the principle of conservation of moment of momentum. The motion is referred to the invariable plane of the system, and however the planet and satellite may interact on one another, the resultant m. of m. must remain constant in direction and magnitude. Hence if we draw a parallelogram of which the diagonal is h (the resultant m. of m. of the system), and of which the sides are n and x, inclined respectively to the diagonal at the angles i and j, we see at once that

$$\frac{\sin i}{\sin j}=\frac{x}{n}.$$

If i and j be treated as small this reduces to (7).

Again the consideration of this parallelogram shows that

$$h^2=n^2+x^2+2nx\cos(i+j),$$

which expresses the constancy of moment of momentum. If the squares and higher powers of $i+j$ be neglected, this becomes

$$h=n+x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Equation (8) may also be obtained by observing that $dn/dt+dx/dt=0$, and therefore on integration $n+x$ is constant. It is obvious from the principle of m. of m. that the planet's equator and the plane of the satellite's orbit have a common node on the invariable plane of the system.

If we divide equations (4) and (6) by (3), we have the following results:—

$$\frac{1}{i}\frac{di}{dx}=\frac{1}{2n}\left(1+\frac{j}{i}\right)\frac{n-2\Omega}{n-\Omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

$$\frac{1}{e}\frac{de}{dx}=\frac{1}{2x}\frac{11n-18\Omega}{n-\Omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

But from (7) and (8)

$$1+\frac{j}{i}=1+\frac{n}{x}=\frac{h}{x},$$

also $\Omega=x^{-3}$, and $n=h-x$.

Hence (9) and (10) may be written

$$\left.\begin{aligned}\frac{d}{dx}\log i&=\frac{h}{2}\cdot\frac{1}{x(h-x)}\cdot\frac{x^3(h-x)-2}{x^3(h-x)-1}\\ \frac{d}{dx}\log e&=\frac{1}{2x}\cdot\frac{11x^3(h-x)-18}{x^3(h-x)-1}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11).$$

Now $$\frac{h\{x^3(h-x)-2\}}{2x(h-x)\{x^3(h-x)-1\}}=\frac{h}{x(h-x)}+\frac{h}{2}\frac{x^2}{x^4-hx^3+1}.$$

Therefore $$\frac{d}{dx}\log i=\frac{1}{x}+\frac{1}{h-x}+\frac{h}{2}\frac{x^2}{x^4-hx^3+1} \quad . \quad . \quad . \quad . \quad (12).$$

Also $$\frac{11x^3(h-x)-18}{2x\{x^3(h-x)-1\}}=\frac{9}{x}-\frac{7}{2}\cdot\frac{x^2(x-h)}{x^4-hx^3+1}.$$

Therefore $$\frac{d}{dx}\log e=\frac{9}{x}-\frac{7}{2}\cdot\frac{x^2(x-h)}{x^4-hx^3+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These two equations are integrable as they stand, except as regard the last term in each of them.

It was shown in a previous paper that the whole energy of the system, both kinetic and potential, was equal to $\frac{1}{2}\,[n^2-x^{-2}]$.*

Then integrating (12) and (13), and writing down (7) and (8) again, and the expression for the energy, we have the following equations, which give the variations of the elements of the system in terms of the square root of the satellite's distance, and independently of the time.

$$\left.\begin{aligned}
\log i&=\log\frac{x}{h-x}+\tfrac{1}{2}h\int\frac{x^2dx}{x^4-hx^3+1}+\text{const.}\\
\log e&=\log x^9-\tfrac{7}{2}\int\frac{x^2(x-h)dx}{x^4-hx^3+1}+\text{const.}\\
j&=\frac{h-x}{x}i.\\
n&=h-x.\\
2E&=(h-x)^2-\frac{1}{x^2}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (14).$$

When the integration of these equations is completed, we shall have the means of tracing the history of a fluid planet of small viscosity, attended by a single satellite, when the system is started with any given moment of momentum h, and with any mean distance and (small) inclination and (small) eccentricity of the satellite's orbit, and (small) obliquity of the planet's equator. It may be remarked that h is to be taken as essentially positive, because the sign of h merely depends on the convention which we choose to adopt as to positive and negative rotations.

These equations do not involve the time, but it will be shown later how the time may be also found as a function of x. It is not, however, necessary to find the expression for the time in order to know the sequence of events in the history of the system.

Since the fluid which forms the planet is subject to friction, therefore the system is non-conservative of energy, and therefore x must change in such a way that E may diminish.

If the expression for E be illustrated by a curve in which E is the vertical ordinate and x the horizontal abscissa, then any point on this "curve of energy" may be taken to represent one configuration of the system, as far as regards the mean distance of the satellite.

* "The Secular Effects of Tidal Friction," &c., "Proc. Roy. Soc.," No. 197, 1879, eq. (4).

Then such a point must always slide down a slope of energy, and we shall see which way x must vary for any given configuration. This consideration will enable us to determine the sequence of events, when we come to consider the expressions for i, e, j, n in terms of x.

We have now to consider the further steps towards the complete solution of the problem.

The only difficulty remaining is the integration of the two expressions in the first and second of (14). From the forms of the expressions to be integrated, it is clear that they must be split up into partial fractions. The forms which these fractions will assume will of course depend on the nature of the roots of the equation $x^4-hx^3+1=0$.

Some of the properties of this biquadratic were discussed in a previous paper, but it will now be necessary to consider the subject in more detail.

It will be found by Ferrari's method that

$$x^4-hx^3+1=\left\{x^2+2x\frac{\lambda^{\frac{3}{2}}-h}{4}+\frac{\lambda^{\frac{3}{2}}-h}{2\lambda^{\frac{1}{2}}}\right\}\left\{x^2-2x\frac{\lambda^{\frac{3}{2}}+h}{4}+\frac{\lambda^{\frac{3}{2}}+h}{2\lambda^{\frac{1}{2}}}\right\}$$

where $\lambda^3-4\lambda-h^2=0$.

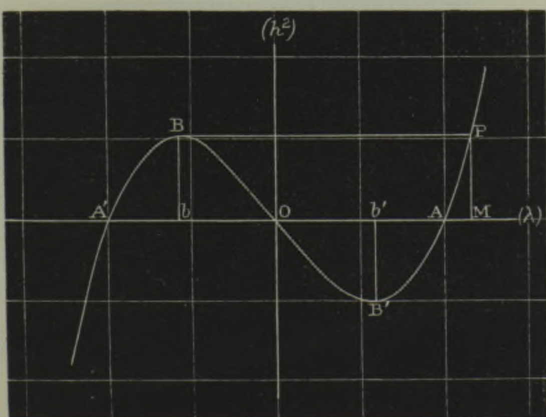
By using the property $(\lambda^{\frac{3}{2}}-h)\,(\lambda^{\frac{3}{2}}+h)=4\lambda$, this expression may be written in the form

$$[\{x+\tfrac{1}{4}(\lambda^{\frac{3}{2}}-h)\}^2+\{\tfrac{1}{4}(\lambda^{\frac{3}{2}}-h)\sqrt{1+2h\lambda^{-\frac{3}{2}}}\}^2]\times$$
$$[\{x-\tfrac{1}{4}(\lambda^{\frac{3}{2}}+h)\}^2+\{\tfrac{1}{4}(\lambda^{\frac{3}{2}}+h)\sqrt{1-2h\lambda^{-\frac{3}{2}}}\}^2],$$

Which is of course equivalent to finding all the roots of the biquadratic in terms of h and λ.

Now let a curve be drawn of which h^2 is the ordinate (negative

FIG. 1.



The ordinates are drawn to one-third of the scale to which the abscissæ are drawn.

values of h^2 being admissible) and λ the abscissa; it is shown in fig. 1. Its equation is $h^2=\lambda(\lambda^2-4)$.

It is obvious that $OA=OA'=2$.

The maximum and minimum values of h^2 (viz., Bb, $B'b'$) are given by $3\lambda^2=4$ or $\lambda=\pm 2/3$.

Then $Bb=B'b'=-2^3/3^{\frac{3}{2}}+4\,.\,2/3^{\frac{1}{2}}=(4/3^{\frac{3}{4}})^2$.

Since in the cubic, on which the solution of the biquadratic depends, h^2 is necessarily positive, it follows that if h be greater than $4/3^{\frac{3}{4}}$ the cubic has one real positive root greater than OM, and if h be less than $4/3$, it has two real negative roots lying between O and OA′, and one real positive root lying between OA and OM.

To find OM we observe that since h^2 is equal to $(4/3^{\frac{3}{4}})^2$, and since the root of $\lambda^3-4\lambda-h^2=0$ which is equal to $-2/3^{\frac{1}{2}}$ is repeated twice, therefore, if ϵ be the third root (or OM) we must have

$$\left(\lambda+\frac{2}{3^{\frac{1}{2}}}\right)^2(\lambda-\epsilon)=\lambda^3-4\lambda-\left(\frac{4}{3}\right)^2,$$

whence $(2/3^{\frac{1}{2}})^2\epsilon=(4/3^{\frac{3}{4}})^2$, and ϵ or $OM=4/3^{\frac{1}{2}}$.

Now $OA=2$; hence, if h be less than $4/3^{\frac{3}{4}}$, the cubic has a positive root between 2 and $4/3^{\frac{1}{2}}$, and if h be greater than $4/3^{\frac{3}{4}}$, the cubic has a positive root between $4/3^{\frac{1}{2}}$ and infinity.

It will only be necessary to consider the positive root of the cubic.

Now suppose h to be greater than $4/3^{\frac{3}{4}}$.

Then it has just been shown that λ is greater than $4/3^{\frac{1}{2}}$, and hence (λ being positive) $3\lambda^3$ is greater than 16λ, or $4(\lambda^3-4\lambda)$ greater than λ^3, or $4h^2$ greater than λ^3, or $2h\lambda^{-\frac{3}{2}}$ greater than unity.

Therefore $\{\frac{1}{4}(\lambda^{\frac{3}{2}}+h)\sqrt{1-2h\lambda^{-\frac{3}{2}}}\}^2=-\{\frac{1}{4}(\lambda^{\frac{3}{2}}+h)\sqrt{2h\lambda^{-\frac{3}{2}}-1}\}^2$.

Thus the biquadratic has two real roots, which we may call a and b.

Then
$$a=\tfrac{1}{4}(\lambda^{\frac{3}{2}}+h)[1+\sqrt{2h\lambda^{-\frac{3}{2}}-1}],$$
$$b=\tfrac{1}{4}(\lambda^{\frac{3}{2}}+h)[1-\sqrt{2h\lambda^{-\frac{3}{2}}-1}].$$

It will now be proved that a is greater and b less than $\frac{3}{4}h$.

Now
$$a> \text{ or } <\tfrac{3}{4}h,$$
$$\text{as } (\lambda^{\frac{3}{2}}+h)[1+\sqrt{2h\lambda^{-\frac{3}{2}}-1}]> \text{ or } <3h,$$
$$\text{as } \frac{(\lambda^{\frac{3}{2}}+h)}{\lambda^{\frac{3}{4}}}\sqrt{2h-\lambda^{\frac{3}{2}}}> \text{ or } <2h-\lambda^{\frac{3}{2}},$$
$$\text{as } \lambda^{\frac{3}{2}}+h> \text{ or } <\lambda^{\frac{3}{4}}\sqrt{2h-\lambda^{\frac{3}{2}}},$$
$$\text{as } \lambda^3+2h\lambda^{\frac{3}{2}}+h^2> \text{ or } <2h\lambda^{\frac{3}{2}}-\lambda^3,$$
$$\text{as } 2\lambda^3+h^2> \text{ or } <0.$$

Since the left hand side is essentially positive, a is greater than $\frac{3}{4}h$.

Again
$$b > \text{ or } < \tfrac{3}{4}h,$$
$$\text{as } (\lambda^{\frac{3}{2}}+h)[1-\sqrt{2h\lambda^{-\frac{3}{2}}-1}] > \text{ or } < 3h,$$
$$\text{as } -\frac{(\lambda^{\frac{3}{2}}+h)}{\lambda^{\frac{3}{4}}}\sqrt{2h-\lambda^{\frac{3}{2}}} > \text{ or } < 2h-\lambda^{\frac{3}{2}}.$$

Since the left-hand side is negative and the right positive, the left is less than the right, and therefore b is less than $\tfrac{3}{4}h$.

If, therefore, h be greater than $4/3^{\frac{3}{2}}$, we may write
$$x^4-hx^3+1=(x-a)(x-b)[(x-\alpha)^2+\beta^2],$$
where $a-\tfrac{3}{4}h$, $\tfrac{3}{4}h-b$ are positive, and where α is negative.

We now turn to the other case and suppose h less than $4/3^{\frac{3}{2}}$. All the roots of the biquadratic are now imaginary, and we may put
$$x^4-hx^3+1=[(x-\alpha)^2+\beta^2][(x-\gamma)^2+\delta^2].$$

If α be taken as $-\tfrac{1}{4}(\lambda^{\frac{3}{2}}-h)$, then γ is $\tfrac{1}{4}(\lambda^{\frac{3}{2}}+h)$.

Then it only remains to prove that γ is greater than $\tfrac{3}{4}h$.

Now
$$\gamma > \text{ or } < \tfrac{3}{4}h,$$
$$\text{as } \lambda^{\frac{3}{2}} > \text{ or } < 2h,$$
$$\text{as } \lambda^3 > \text{ or } < 4h^2=4(\lambda^3-4\lambda),$$
$$\text{as } 16 > \text{ or } < 3\lambda^2,$$
$$\text{as } 4/3^{\frac{1}{2}} > \text{ or } < \lambda,$$
but it has been already shown that in this case, λ is less than $4/3^{\frac{1}{2}}$, wherefore γ is greater than $\tfrac{3}{4}h$.

We may now proceed to the required integrations.

First case where h is greater than $4/3^{\frac{3}{2}}$.

Let
$$x^4-hx^3+1=(x-a)(x-b)[(x-\alpha)^2+\beta^2],$$
so that the roots are a, b, $\alpha\pm\beta\iota$.

Also let a be the root which is greater than $\tfrac{3}{4}h$, b that which is less, and let
$$a=a_1+\tfrac{3}{4}h, \qquad b=\tfrac{3}{4}h-b_1, \qquad \alpha=\tfrac{3}{4}h-\alpha_1.$$

To find the expression for i we have to integrate $\dfrac{x^2}{x^4-hx^3+1}$.

Let $f(x)=(x-a)\psi(x)$, and let $x^2/f(x)=A/(x-a)+\phi(x)/\psi(x)$.

Then
$$x^2(x-a)=Af(x)+(x-a)^2\phi(x).$$
Hence
$$A=a^2/f'(a).$$
If, therefore, $f(x)=x^4-hx^3+1$, $A=1/(4a-3h)=1/4a_1$.

Thus the partial fractions corresponding to the roots a and b are

$$\frac{1}{4a_1}\frac{1}{x-a}-\frac{1}{4b_1}\frac{1}{x-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

If the pair of fractions corresponding to the roots $\alpha \pm \beta\iota$ be formed and added together, we find

$$\frac{1}{2(\alpha_1^2+\beta^2)}\frac{-\alpha_1(x-\alpha)+\beta^2}{[(x-\alpha)^2+\beta^2]} \quad . \quad . \quad . \quad . \quad . \quad (16).$$

The sum of (15) and (16) is equal to $\frac{x^2}{x^4-hx^3+1}$, and

$$\int\frac{x^2dx}{x^4-hx^3+1}=\frac{1}{4a_1}\log(x\backsim a)-\frac{1}{4b_1}\log(x\backsim b)-\frac{\alpha_1}{4(\alpha_1^2+\beta^2)}\log[(x-\alpha)^2+\beta^2]$$

$$+\frac{\beta}{2(\alpha_1^2+\beta^2)}\arctan\frac{x-\alpha}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Substituting in the first of (14) we have

$$i=A\frac{x}{h-x}\cdot\frac{(x\backsim a)^{\frac{h}{8a_1}}\exp.\left[\frac{h\beta}{4(\alpha_1^2+\beta^2)}\arctan\frac{x-\alpha}{\beta}\right]}{(x\backsim b)^{\frac{h}{8b_1}}\left[(x-\alpha)^2+\beta^2\right]^{\frac{h\alpha_1}{8(\alpha_1^2+\beta^2)}}} \quad . \quad . \quad (18).$$

where A is a constant to be determined by the value of i, which corresponds with a particular value of x.

From the third of (14) we see that by omitting the factor $x/(h-x)$ from the above, we obtain the expression for j.

To find the expression for e we have to integrate $\frac{x^2(x-h)}{x^4-hx^3+1}$.

Now $$x^2(x-h)=\tfrac{1}{4}(4x^3-3hx^2)-\tfrac{1}{4}hx^2,$$
and therefore

$$\int\frac{x^2(x-h)dx}{x^4-hx^3+1}=\tfrac{1}{4}\log(x^4-hx^3+1)-\tfrac{1}{4}h\int\frac{x^2dx}{x^4-hx^3+1}.$$

The integral remaining on the right hand has been already determined in (17). Then substituting in the second of (14), we have

$$e=\frac{Bx^9}{(x^4-hx^3+1)^{\frac{7}{4}}}\left(\frac{(x\backsim a)^{\frac{h}{8a_1}}\exp.\left[\frac{h\beta}{4(\alpha_1^2+\beta^2)}\arctan\frac{x-\alpha}{\beta}\right]}{(x\backsim b)^{\frac{h}{8b_1}}\left[(x-\alpha)^2+\beta^2\right]^{\frac{h\alpha_1}{8(\alpha_1^2+\beta^2)}}}\right)^{\frac{7}{4}} \quad . \quad (19).$$

where B is a constant to be determined by the value of e, corresponding to some particular value of x.

From this equation we get the curious relationship

$$e=\frac{B}{A^{\frac{7}{4}}}\cdot\frac{x^9}{(x^4-hx^3+1)^{\frac{7}{8}}}j^{\frac{7}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

This last result will obviously be equally true even if all the roots of $x^4-hx^3+1=0$ are imaginary.

In the present case the complete solution of the problem is comprised in the following equations:—

$$\left.\begin{aligned}
j&=A\frac{(x\backsim \mathrm{a})^{\frac{h}{8\mathrm{a}_1}}\exp.\left[\frac{h\beta}{4(\alpha_1{}^2+\beta^2)}\arc\tan\frac{x-\alpha}{\beta}\right]}{(x\backsim \mathrm{b})^{\frac{h}{8\mathrm{b}_1}}[(x-\alpha)^2+\beta^2]^{\frac{h\alpha_1}{8(\alpha_1{}^2+\beta^2)}}}\\
i&=\frac{x}{h-x}j.\\
e&=\frac{B}{A^{\frac{7}{4}}}\frac{x^9}{(x^4-hx^3+1)^{\frac{7}{8}}}j^{\frac{7}{4}}.\\
n&=h-x.\\
2E&=(h-x)^2-\frac{1}{x^2}.
\end{aligned}\right\} \quad . \quad . \quad (21).$$

It is obvious that the system can never degrade in such a way that x should pass through one of the roots of the biquadratic $x^4-hx^3+1=0$. Hence the solution is divided into three fields, viz., (i) $x=+\infty$ to $x=\mathrm{a}$; here we must write $x-\mathrm{a}$, $x-\mathrm{b}$ for the $x\backsim\mathrm{a}$, $x\backsim\mathrm{b}$ in the above solution; (ii) $x=\mathrm{a}$ to $x=\mathrm{b}$; here we must write $\mathrm{a}-x$, $x-\mathrm{b}$ (this is the part which has most interest in application to actual planets and satellites); (iii) $x=\mathrm{b}$ to $x=-\infty$; here we must write $\mathrm{a}-x$, $\mathrm{b}-x$. When x is negative the physical meaning is that the revolution of the satellite is adverse to the planet's rotation.

By referring to (4) and (6), we see that i must be a maximum or minimum when $n=2\Omega$, and e a maximum or minimum when $n=\frac{18}{11}\Omega$. Hence the corresponding values of x are the roots of the equations $x^4-hx^3+2=0$, and $x^4-hx^3+\frac{18}{11}=0$ respectively.

Since

$$\frac{x^2}{x^4-hx^3+1}=\frac{1}{4\mathrm{a}_1}\frac{1}{x-\mathrm{a}}-\frac{1}{4\mathrm{b}_1}\frac{1}{x-\mathrm{b}}+\frac{1}{2(\alpha_1{}^2+\beta^2)}\frac{-\alpha_1(x-\alpha)+\beta^2}{[(x-\alpha)^2+\beta^2]}.$$

Therefore

$$x^2=\frac{1}{4\mathrm{a}_1}(x-\mathrm{b})[(x-\alpha)^2+\beta^2]-\frac{1}{4\mathrm{b}_1}(x-\mathrm{a})[(x-\alpha)^2+\beta^2]$$
$$+\frac{1}{2(\alpha_1{}^2+\beta^2)}[-\alpha_1(x-\alpha)+\beta^2](x-\mathrm{a})(x-\mathrm{b}).$$

Hence the coefficient of x^3 on the right-hand side must be zero, and therefore $\frac{1}{4a_1}-\frac{1}{4b_1}-\frac{\alpha_1}{2(\alpha_1^2+\beta^2)}=0$.

And $\frac{h}{8a_1}=\frac{h}{8b_1}+\frac{h\alpha_1}{4(\alpha_1^2+\beta^2)}$.

Now when $x=+\infty$, $\text{arc tan}\frac{x-\alpha}{\beta}=\frac{1}{2}\pi$, and when $x=-\infty$, it is equal to $-\frac{1}{2}\pi$.

Hence when $x=\pm\infty$, $j=A\exp.\,[\pm\pi h\beta/8(\alpha_1^2+\beta^2)]$, $i=-j$; the upper sign being taken for $+\infty$ and the lower for $-\infty$.

Then since j tends to become constant when $x=\pm\infty$, and since $9-\frac{7}{2}=\frac{11}{2}$, therefore when x is very large e tends to vary as $x^{\frac{11}{2}}$.

If x be very small j has a finite value, and i varies as x, and e varies as x^9.

j, i, and e all become infinite when $x=$b, and i also becomes infinite when $x=h$.

This analytical solution is so complex that it is not easy to understand its physical meaning; a geometrical illustration will, however, make it intelligible.

The method adopted for this end is to draw a series of curves, the points on which have x as abscissa and i, j, e, n, E as ordinates. The figure would hardly be intelligible if all the curves were drawn at once, and therefore a separate figure is drawn for i, j, and e; but in each figure the straight line which represents n is drawn, and the energy curve is also introduced in order to determine which way the figure is to be read. The zero of energy is of course arbitrary, and therefore the origin of the energy curve is in each case shifted along the vertical axis, in such a way that the energy curve may clash as little as possible with the others.

It is not very easy to select a value of h which shall be suitable for drawing these curves within a moderate compass, but after some consideration I chose $h=2{\cdot}6$, and figs. 2, 3, and 4 are drawn to illustrate this value of h. If the cubic $\lambda^3-4\lambda-(2{\cdot}6)^2=0$, be solved by Cardan's method, it will be found that $\lambda=2{\cdot}5741$, and using this value in the formula for the roots of the biquadratic we have

$$x^4-2{\cdot}6\,x^3+1=(x-2{\cdot}539)(x-{\cdot}826)[(x+{\cdot}382)^2+({\cdot}575)^2].$$

Hence $\text{a}=2{\cdot}539$, $\text{b}={\cdot}826$, $\alpha=-{\cdot}382$, $\beta={\cdot}575$, $\frac{3}{4}h=1{\cdot}95$, and $4a_1=2{\cdot}356$, $4b_1=4{\cdot}496$, $\alpha_1=2{\cdot}332$, $\alpha_1^2+\beta^2=5{\cdot}771$.

Then we have

$$\left.\begin{aligned}
j&=A\frac{(2{\cdot}539\backsim x)^{\cdot 552}\exp.\left[\cdot 062\text{ arc tan }(1{\cdot}740x+\cdot 665)\right]}{(x\backsim\cdot 826)^{\cdot 289}(x^2+\cdot 765x+\cdot 477)^{\cdot 1314}}.\\
i&=\frac{x}{2{\cdot}6-x}j.\\
e&=\frac{B}{A^{\frac{7}{4}}}\frac{x^9}{(x^4-2{\cdot}6x^3+1)^{\frac{1}{4}}}j^{\frac{7}{4}}.\\
n&=2{\cdot}6-x.\\
2E&=(2{\cdot}6-x)^2-\frac{1}{x^2}.
\end{aligned}\right\}\quad\ldots\quad(22).$$

The maximum and minimum values of i are given by the roots of the equation $x^4-2{\cdot}6x^3+2=0$, viz., $x=2{\cdot}467$ and $x=1{\cdot}103$. The maximum and minimum values of e are given by the roots of the equation $x^4-2{\cdot}6x+\frac{18}{11}=0$, viz., $x=2{\cdot}495$ and $x=1{\cdot}0095$. The horizontal asymptotes for i/A and j/A are at distances from the axis of x equal to exp. $(\cdot 062\times\frac{1}{2}\pi)$ and exp. $(-\cdot 062\times\frac{1}{2}\pi)$, which are equal to 1·102 and ·908 respectively.

Fig. 2 shows the curve illustrating the changes of i, the obliquity of the equator to the invariable plane.

The asymptotes are indicated by broken lines; that at A is given by $x=\cdot 826$, and is the ordinate of maximum energy; that at B is given by $x=2{\cdot}6$, and gives the configuration of the system for which the planet has no rotation. The point C is given by $x=2{\cdot}539$, and lies on the ordinate of minimum energy. Geometrically the curve is divided into three parts by the vertical asymptotes, but it is further divided physically.

The curve of energy has four slopes, and since the energy must degrade, there are four methods in which the system may change, according to the way in which it was started. The arrows marked on the curve of obliquity show the direction in which the curve must be read.

Since none of these four methods can ever pass into another, this figure really contains four figures, and the following parts of the figure are quite independent of one another, viz.: (i) from $-\infty$ to O; (ii) from A to O; (iii) from A to C; (iv) from $+\infty$ to C. The figures 3 and 4 are similarly in reality four figures combined. For each of these parts the constant A must be chosen with appropriate sign; but in order to permit the curves in fig. 2 to be geometrically continuous the obliquity is allowed to change sign.

The actual numerical interpretation of this figure depends on the value of A. Thus if for any value of x in any of the four fields the obliquity has an assigned value, then the ordinate corresponding to that value of x will give a scale of obliquity from which all the other ordinates within that field may be estimated.

FIG. 2.

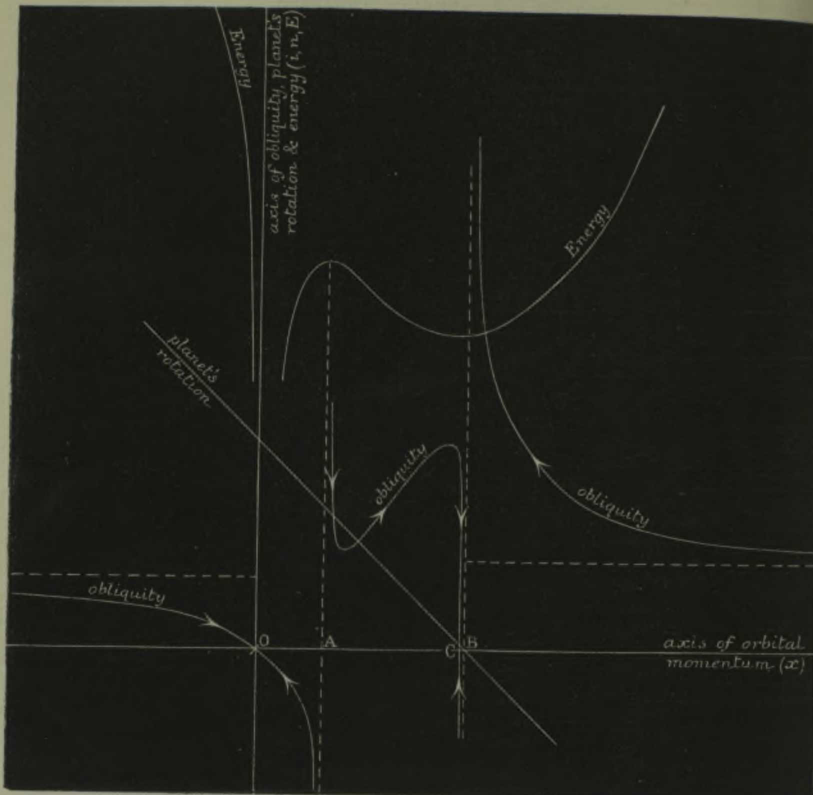


Diagram for Obliquity of Planet's Equator.—First case.

As a special example of this we see that, if the obliquity be zero at any point, a consideration of the curve will determine whether zero obliquity be dynamically stable or not; for if the arrows on the curve of obliquity be approaching the axis of x, zero obliquity is dynamically stable, and if receding from the axis of x, dynamically unstable.

Hence from $x=+\infty$ to B, zero obliquity is dynamically unstable, from $-\infty$ to O and A to O dynamically stable, and from A to B, first stable, then unstable, and finally stable.

The infinite value of the obliquity at the point B has a peculiar significance, for at B the planet has no rotation, and being thus free from what Sir William Thomson calls "gyroscopic domination," the obliquity changes with infinite ease. In fact at B the term equator loses its meaning. The infinite value at A has a different meaning. The configuration A is one of maximum energy and of dynamical equilibrium, but is unstable as regards mean distance and planetary rotation; at this point the system changes infinitely slowly as regards ime, and therefore the infinite value of the obliquity does not indicate n infinite rate of change of obliquity. In fact if we put $n=\Omega$ in (1) ve see that $di/idt=-\frac{1}{4}(\tau^2/\mathfrak{g})\sin 4f$. However, to consider this case dequately we should have to take into account the obliquity in the quations for dn/dt and dx/dt, because the principal semi-diurnal tide anishes when $n=\Omega$.

Similarly at the minimum of energy the system changes infinitely lowly, and thus the obliquity would take an infinite time to vanish.

We may now state the physical meaning of fig. 2, and this interpre-ation may be compared with a similar interpretation in the paper on "The secular effects of tidal friction," above referred to.

A fluid planet of small viscosity is attended by a single satellite, and he system is started with an amount of positive moment of momentum which is greater than $4/3^{\frac{3}{4}}$, with our present units of length, mass and ime.

The part of the figure on the negative side of the origin indicates a negative revolution of the satellite and a positive rotation of the planet, but the m. of m. of planetary rotation is greater (by an amount h) than the m. of m. of orbital motion. Then the satellite approaches the planet and ultimately falls into it, and the obliquity always diminishes slowly. The part from O to A indicates positive rotation of both parts of the system, but the satellite is very close to the planet and revolves round the planet quicker than the planet rotates, as in the case of the inner satellite of Mars. Here again the satellite approaches and ultimately falls in, and the obliquity always diminishes.

The part from A to C indicates positive rotation of both parts, but the satellite revolves slower than the planet rotates. This is the case which has most interest for application to the solar system. The satellite recedes from the planet, and the system ceases its changes when the satellite and planet revolve slowly as parts of a rigid body—that is to say, when the energy is a minimum. The obliquity first decreases, then increases to a maximum, and ultimately decreases to zero.*

The part from infinity to C indicates a positive revolution of the satellite, and from infinity to B a negative rotation of the planet, but from B to C a positive rotation of the planet, which is slower than the revolution of the satellite. In either of these cases the satellite approaches the planet, but the changes cease when the satellite and planet move slowly round as parts of a rigid body—that is to say, when the energy is a minimum. If the rotation of the planet be positive, the obliquity diminishes, if negative it increases. If the rotation of the planet be *nil*, the term obliquity ceases to have any meaning, since there is no longer an equator.

* According to the present theory, the moon, considered as being attended by the earth as a satellite, has gone through these changes.

Fig. 3 illustrates the changes of inclination of the satellite's orbi and may be interpreted in the same way as fig. 2. It appears from th part of the figure for which x is negative, that if the revolution of th satellite be negative, and the rotation of the planet positive, but th

FIG. 3.

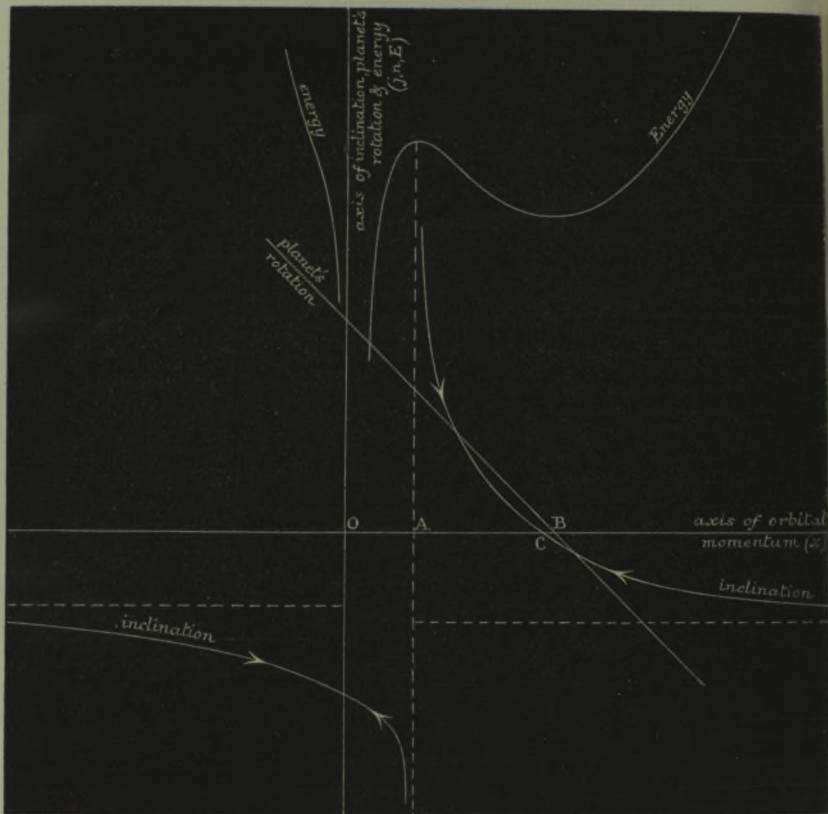


Diagram for Inclination of Satellite's Orbit.—First case.

m. of m. of planetary rotation greater than that of orbital motion, then, as the satellite approaches the planet, the inclination of the orbit increases, or zero inclination is dynamically unstable. In every other case the inclination will decrease, or zero inclination is dynamically stable.

This result undergoes an important modification when a second satellite is introduced, as will appear in the unpublished paper.

Fig. 4 shows a similar curve for the eccentricity of the orbit. The variations of the eccentricity are very much larger than than those of the obliquity and inclination, so that it was here necessary to draw the ordinates on a much reduced scale. It was not possible to extend the figure far in either direction, because for large values of x, e varies as a high power of x (viz., $\frac{11}{2}$). The curve presents a resemblance to that of obliquity, for in the field comprised between the two roots of the biquadratic (viz., between A and C) the eccentricity diminishes to a minimum, increases to a maximum, and ultimately vanishes at C.

Fig. 4.

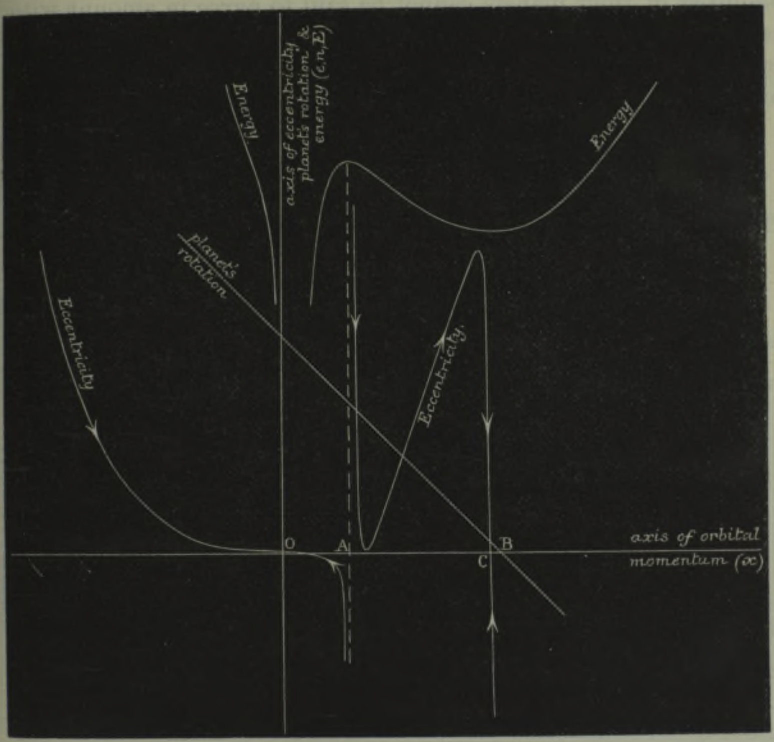


Diagram for Eccentricity of Satellite's Orbit.—First case.

This field represents a positive rotation both of the planet and satellite, but the satellite revolves slower than the planet rotates. This part represents the degradation of the system from the configuration of maximum energy to that of minimum energy, and the satellite recedes from the planet, until the two move round slowly like the parts of a rigid body.

In every other case the eccentricity degrades rapidly, whilst the satellite approaches the planet.

The very rapid rate of variation of the eccentricity, compared with that of the obliquity would lead one to expect that the eccentricity of the orbit of a satellite should become very large in the course of its evolution, whilst the obliquity should not increase to any very large extent. But it must be remembered that we are here only treating a planet of small viscosity, and it will appear, in the unpublished paper above referred to, that the rate of increase or diminution of the eccentricity is very much less rapid (per unit increase of x) if the viscosity be not small, whilst the rate of increase or diminution of obliquity (per unit increase of x) is slightly increased with increase of viscosity. Thus the observed eccentricities of the orbits of satellites and of obliquities of their planets cannot be said to agree in amount with the theory that the planets were primitively fluids of small viscosity, though I believe they *do* agree with the theory that the planets were fluids or quasi-solids of large viscosity.

We now come to the *second case*, where h is less than $4/3^{\frac{3}{4}}$. The biquadratic having no real roots, we may put

$$x^4-hx^3+1=[(x-a)^2+\beta^2][(x-\gamma)^2+\delta^2].$$

It has already been shown that a is negative, and γ greater than $\frac{3}{4}h$.

Let $$a=\tfrac{3}{4}h-a_1, \qquad \gamma=\gamma_1+\tfrac{3}{4}h.$$

Then by inspection of the integral in the first case we see that

$$j=A\frac{[(x-\gamma)^2+\delta^2]^{\frac{h\gamma_1}{8(\gamma_1^2+\delta^2)}}}{[(x-\alpha)^2+\beta^2]^{\frac{h\alpha_1}{8(\alpha_1^2+\beta^2)}}}\times \exp.\left[\frac{h\beta}{4(\alpha_1^2+\beta^2)}\arctan\frac{x-\alpha}{\beta}+\frac{h\delta}{4(\gamma_1^2+\delta^2)}\arctan\frac{x-\gamma}{\delta}\right].$$

The rest of equations (21), which express the other elements in terms of j and x, remain the same as before.

By comparison with the first case, we see that

$$\frac{x^2}{x^4-hx^3+1}=\frac{1}{2(\alpha_1^2+\beta^2)}\frac{-\alpha_1(x-\alpha)+\beta^2}{(x-\alpha)^2+\beta^2}+\frac{1}{2(\gamma_1^2+\delta^2)}\frac{\gamma_1(x-\gamma)+\delta^2}{(x-\gamma)^2+\delta^2}.$$

On multiplying both sides of this identity by x^4-hx^3+1, and equating the coefficients of x^3, we find

$$0=\frac{-\alpha_1}{2(\alpha_1^2+\beta^2)}+\frac{\gamma_1}{2(\gamma_1^2+\delta^2)}.$$

Therefore $$\frac{h\alpha_1}{8(\alpha_1^2+\beta^2)}=\frac{h\gamma_1}{8(\gamma_1^2+\delta^2)}.$$

Thus when x is equal to $\pm\infty$

$$j=A\exp.\left[\pm\frac{\pi h\beta}{8(\alpha_1^2+\beta^2)}\pm\frac{\pi h\delta}{8(\gamma_1^2+\delta^2)}\right],$$

he upper sign being taken for $+\infty$, and the lower for $-\infty$. This xpression gives the horizontal asymptotes for j and i.

In order to illustrate this solution, I chose $h=1$, and found by trigonometrical solution of the cubic $\lambda^3-4\lambda-1=0$, $\lambda=2{\cdot}1149$, and hence

$$\left.\begin{aligned}
j&=A\left(\frac{x^2-2{\cdot}038x+1{\cdot}401}{x^2+1{\cdot}038x+{\cdot}714}\right)^{{\cdot}077}\text{exp.}\,[{\cdot}081\,\text{arc tan}\,(1{\cdot}500x+{\cdot}778)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+{\cdot}346\,\text{arc tan}\,(1{\cdot}659x-1{\cdot}691)].\\
i&=\frac{x}{1-x}j.\\
e&=\frac{B}{A^{\frac{7}{4}}}\frac{x^9}{(x^4-x^3+1)^{\frac{7}{8}}}j^{\frac{7}{4}}.\\
n&=1-x.\\
2E&=(1-x)^2-\frac{1}{x^2}.
\end{aligned}\right\}\quad(23)$$

When $x=+\infty$, $j/A=1{\cdot}956=-i/A$, and when $x=-\infty$, $j/A={\cdot}512=-i/A$.

FIG. 5.

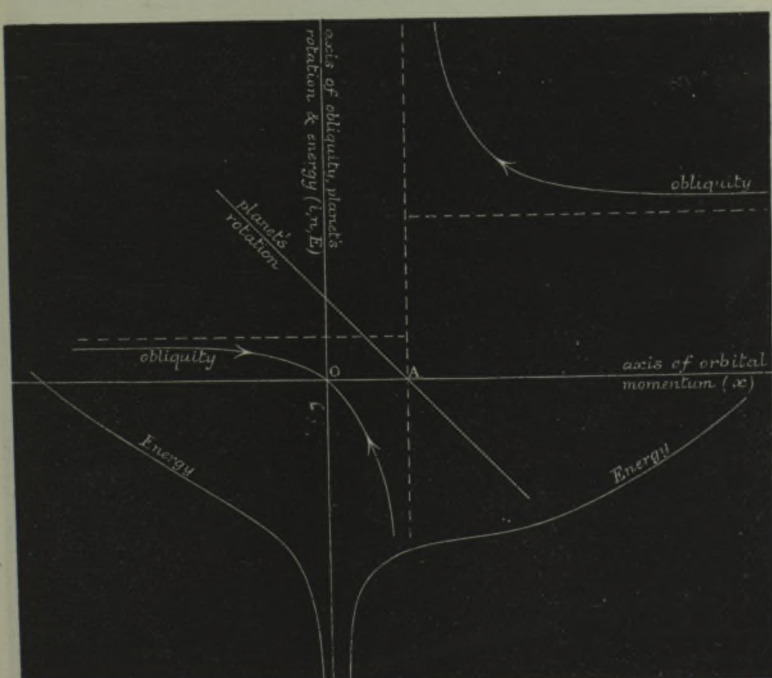


Diagram for Obliquity of Planet's Equator.—Second case.

These solutions are illustrated as in the previous case by the three figures 5, 6, 7. There are here only two slopes of energy, and hence these figures each of them only contain two separate figures.

Fig. 5 illustrates the changes of i, the obliquity of the equator to the invariable plane.

In this figure there is only one vertical asymptote, viz., that corresponding to $x=1$. For this value of x the planet has no rotation, is free from "gyroscopic domination," and the term equator loses its meaning.

The figure shows that if the rotation of the planet be negative, but the m. of m. of planetary rotation less than that of orbital motion, then the obliquity increases, whilst the satellite approaches the planet.

This increase of obliquity only continues so long as the rotation of the planet is negative. The rotation becomes positive after a time, and the obliquity then diminishes, whilst the satellite falls into the planet. In the corresponding part of fig. 2 the satellite did not fall into the planet, but the two finally moved slowly round together as the parts of a rigid body.

If the revolution of the satellite be negative, and the rotation of the planet positive, but the m. of m. of rotation greater than that of revolution, the obliquity always diminishes as the satellite falls in to the planet.

Fig. 6.

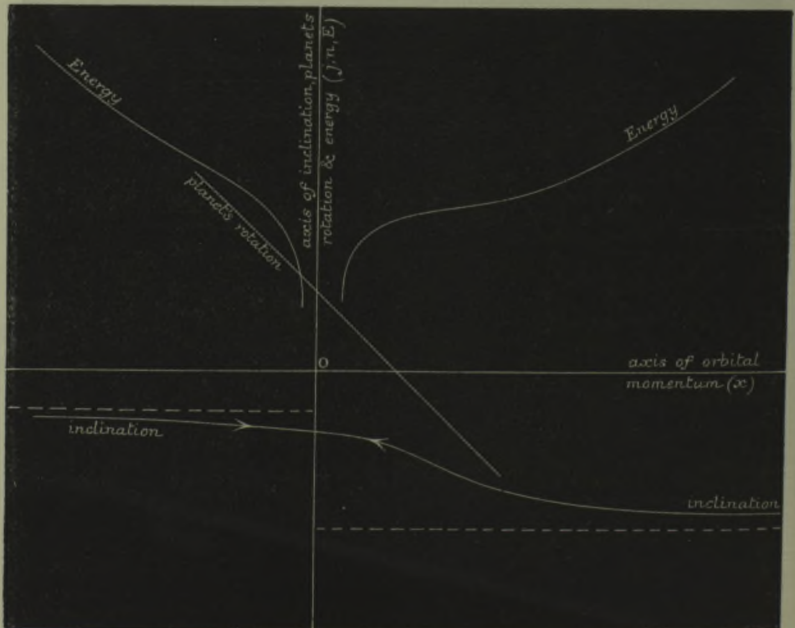


Diagram for Inclination of Satellite's Orbit.—Second case.

Figs. 2 and 5 only differ in the fact that in the one there is a true maximum and a true minimum of obliquity and energy, and in the other there is not so. In fact, if we annihilate the part between the vertical asymptotes of fig. 2 we get fig. 5.

Fig. 6 illustrates the changes of inclination of the orbit. It does not possess very much interest, since it simply shows that however the system be started with positive revolution of the satellite, whether the rotation of the planet be positive or not, the inclination of the orbit slightly diminishes as the satellite falls in.

And however the system be started with negative revolution of the satellite, and therefore necessarily positive rotation of the planet, the inclination of the orbit slightly increases. Fig. 6 again corresponds to fig. 3, if in the latter the part lying between the maximum and minimum of energy be annihilated.

FIG. 7.

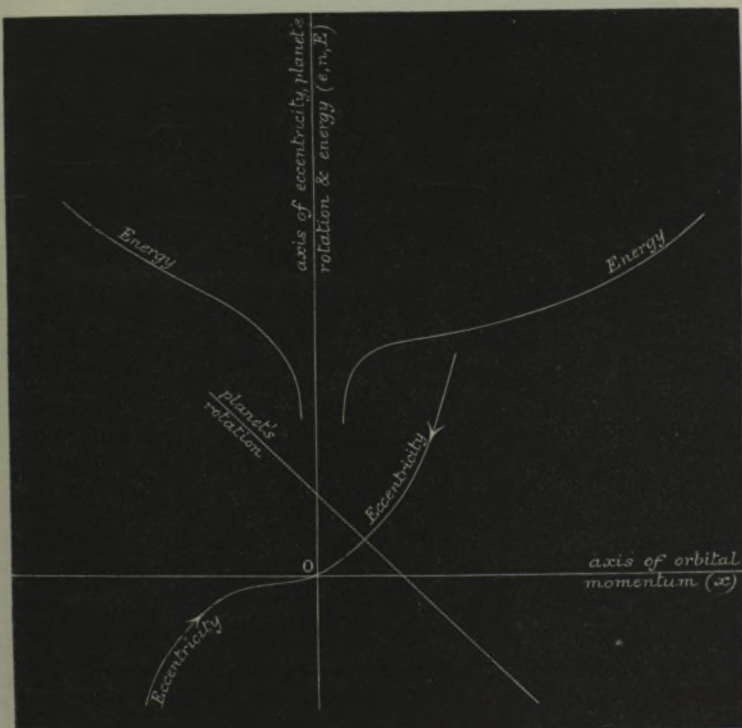


Diagram for Eccentricity of Satellite's Orbit.—Second case.

Fig. 7 illustrates the changes of eccentricity, and shows that it always diminishes rapidly however the system is started, as the satellite falls into the planet. This figure again corresponds with fig. 4, if in the latter the parts between the maximum and minimum of energy l annihilated.

These three figures may be interpreted as giving the various stabiliti and instabilities of the system, just as was done in the first case.

The solution of the problem, which has been given and discusse above, gives merely the sequence of events, and does not show the rat at which the changes in the system take place. It will now be show how the time may be found as a function of x.

Consider the equation

$$\frac{dx}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f\left(1-\frac{\Omega}{n}\right),$$

f is here the angle of lag of the sidereal semi-diurnal tide of speed $2n$ then by the theory of the tides of a viscous spheroid, $\tan 2f=2n/\mathfrak{p}$ where $\mathfrak{p}$ is a certain function of the radius of the planet and its density and which varies inversely as the coefficient of viscosity of the spheroid.*

Since by hypothesis the viscosity is small, f is a small angle, so that $\sin 4f$ may be taken as equal to $2\tan 2f$. Thus, $\sin 4f/n$ is a constant, depending on the dimensions, density, and viscosity of the planet.

It has already been shown that τ^2 varies as x^{-12}, and $\mathfrak{g}$ is a constant, which depends only on the density of the planet. Hence, the above equation may be written

$$x^{12}\frac{dx}{dt}=\text{K}(n-\Omega),$$

where K is a certain constant, which it is immaterial at present to evaluate precisely.

Since $n=h-x$ and $\Omega=x^{-3}$, we have

$$\text{K}dt=\frac{-x^{15}dx}{x^4-hx^3+1},$$

or

$$\text{K}t=-\int\frac{x^{15}dx}{x^4-hx^3+1}+\text{a const.}$$

The determination of this integral presents no difficulty, but the analytical expression for the result is very long, and it does not at present seem worth while to give the result. The actual scale of time in years will depend on the value of K, and this is a subject of no interest at present.

It will, however, be possible to give an idea of the rate of change of the system without actually performing the integration. This may

* "On the Bodily Tides of Viscous and semi-elastic Spheroids," &c. "Phil. Trans.," Part I, 1879, p. 13, § 5.

e done by drawing a curve in which the ordinates are proportional o dt/dx, and the abscissæ are x. The equation to this curve is then

$$\mathrm{K}\frac{dt}{dx}=\frac{-x^{15}}{x^4-hx^3+1}.$$

The maximum and minimum values (if any) of dt/dx are given by the eal roots of the equation

$$11x^4-12hx^3+15=0.$$

One of such roots will be found to be intermediate between a and b, and the other greater than a.

Fig. 8.

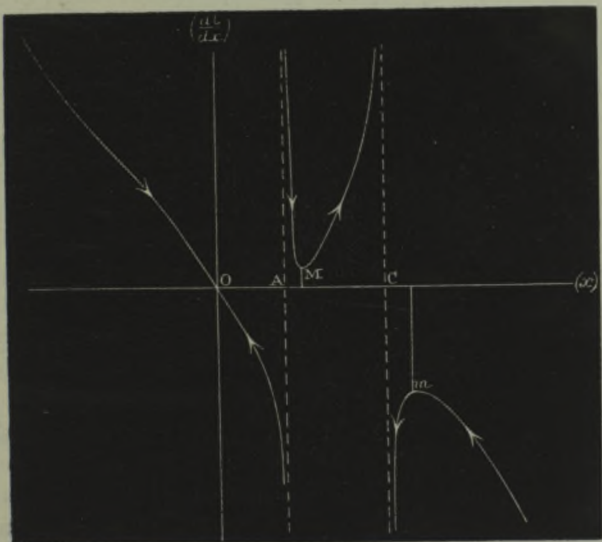


Diagram illustrating the Rate of Change of the System.

Fig. 8 shows the nature of the curve when drawn with the free hand. It was not found possible to draw this figure to scale, because when $h=2{\cdot}6$ it was found that the minimum M was equal to ·85, and could not be made distinguishable from a point on the asymptote A, whilst the minimum m was equal to about 900,000, and could not be made distinguishable from a point on the asymptote C.

The area intercepted between this curve, the axis of x, and any pair of ordinates corresponding to two values of x, will be proportional to the time required to pass from the one configuration to the other.

Where dt/dx is negative, that is to say, when the satellite is falling into the planet, the areas fall below the axis of x. This is clearly necessary in order to have geometrical continuity in the curve.

The figure shows that the rate of alteration in the system becomes very slow when the satellite is far from the planet; this must indeed obviously be the case, because the tidal effects vary as the inverse sixth power of the satellite's mean distance.

V. "On the Modifications of the Spectrum of Potassium which are Effected by the Presence of Phosphoric Acid, and on the Inorganic Bases and Salts which are found in combination with Educts of the Brain." By J. L. W. Thudichum, M.D., F.R.C.P.L. Communicated by John Simon, C.B., F.R.S., &c. Received March 10, 1880.

Among the results of a large investigation on which I have for many years been engaged in regard of the chemistry of the brain, I had been led to conclude that the so-called "protagon" of Oscar Liebreich is not a definite chemical body, but is a variable mixture of several bodies. This conclusion of mine (which agrees with opinions expressed on the same subject by Strecker, Diaconow, and Hoppe-Seyler) was published by me in 1874,* and endeavours to controvert it have since then been made, on several occasions, by Dr. Arthur Gamgee.† Last summer, he brought before the Royal Society‡ his contentions for the chemical individuality of "protagon"; and it fortunately was in my power shortly afterwards to publish evidence, which, I believe, those who will take the trouble to follow it will find quite unanswerable, that Dr. Gamgee's contentions were mistaken.§ Part of my evidence to that effect consisted in showing by quantitative analyses that Dr. Gamgee's so-called "protagon" contains 0·7 per cent. of potassium; secondly, that in connexion with trifling differences in the extraction process, the proportion of potassium in different specimens of "protagon" can be made to range from a trace to 1·6 per cent.; thirdly, that with the variable quantities of potassium the quantities of phosphorus and other ingredients will also vary.

In the last published number, No. 200, p. 111, of the "Proceedings of the Royal Society," I find that Dr. Gamgee has recently brought the question again under notice of the Society, and that, in doing so, he especially rests his case upon the following statement made by his colleague, Professor Roscoe, on the subject of some examinations,

* "Reports of the Medical Officer of the Privy Council and Local Government Board." New Series. No. III.

† "Zeitschrift für Physiol. Chemie," vol. iii, p. 260; "Ber. Deutsch. Chem. Ges.," 1879, &c.

‡ "Proc. Roy. Soc.," vol. xxix, p. 151.

§ "Annals of Chemical Medicine." Edited by J. L. W. Thudichum. Vol. i, p. 254.